Towards a Spectral Theory for Simplicial Complexes

by

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Dissertation submitted in partial fulfillment of the requirements for the degree of
Doctor of Philosophy in the Department of Mathematics
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Abstract

In this dissertation we study combinatorial Hodge Laplacians on simplicial complexes using tools generalized from spectral graph theory. Specifically, we consider generalizations of graph Cheeger numbers and graph random walks. The results in this dissertation can be thought of as the beginnings of a new spectral theory for simplicial complexes and a new theory of high-dimensional expansion.

We first consider new high-dimensional isoperimetric constants. A new Cheeger-type inequality is proved, under certain conditions, between an isoperimetric constant and the smallest eigenvalue of the Laplacian in codimension 0. The proof is similar to the proof of the Cheeger inequality for graphs. Furthermore, a negative result is proved, using the new Cheeger-type inequality and special examples, showing that certain Cheeger-type inequalities cannot hold in codimension 1.

Second, we consider new random walks with killing on the set of oriented simplexes of a certain dimension. We show that there is a systematic way of relating these walks to combinatorial Laplacians such that a certain notion of mixing time is bounded by a spectral gap and such that distributions that are stationary in a certain sense relate to the harmonics of the Laplacian. In addition, we consider the possibility of using these new random walks for semi-supervised learning. An algorithm is devised which generalizes a classic label-propagation algorithm on graphs to simplicial complexes. This new algorithm applies to a new semi-supervised learning problem, one in which the underlying structure to be learned is flow-like.

This dissertation is dedicated to my parents, who always supported my education.

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Acknowledgements

I'd first like to acknowledge my parents, who always supported my education, and my high school math tutor Shail Ganith, under whom I learned much more than I should have. I'd also like to thank my wife, without whom I would not be where I am today, my Uncle Tuck and Aunt Tish, for advice and support, and my two children Julia and Charles, for comedic relief.

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1

Introduction

This dissertation is concerned with the spectral theory of combinatorial Hodge Laplacians on simplicial complexes. Simplicial complexes are discrete, combinatorial objects that have long been used to approximate manifolds and study their topology. The Hodge Laplacians of a smooth manifold, defined using the de Rham complex, are discretely approximated by the combinatorial Laplacians on simplcial complexes. Thus, combinatorial Laplacians lie in the realm of discrete and computational geometry.

In addition, combinatorial Laplacians generalize to higher order the more basic notion of the Laplacian of a graph. Historically, the graph Laplacian has been well-studied and well-applied to modern problems. In 1992, Fan Chung published a book called Spectral Graph Theory that promised, rather poetically, to tell "an intertwined tale of eigenvalues and their use in unlocking a thousand secrets about graphs," and to show " how, through eigenvalues, theory and applications in communications and computer science come together in symbiotic harmony." Indeed, the book lived up to these promises and, more than twenty years later, research in theoretical and applied spectral graph theory is as active as ever.

Compared to the graph Laplacian, combinatorial Laplacians are ill-understood and infrequently applied. The long-running success of spectral graph theory suggests that this need not be the case. It seems reasonable to hope that, just as the graph Laplacian generalizes to higher orders, perhaps spectral graph theory generalizes to higher orders as well. This simple idea has recently gained traction in the literature, and it forms the basis of this dissertation.

The primary difficulty in generalized graph theory to simplicial complexes is the importance of orientation in higher dimensions. When working with graphs the main objects are the vertices, and there is no meaningful notion of orientation for vertices. We meaningfully can ask whether two vertices are connected by an edge, but not whether those two vertices are similarly or dissimilarly oriented. However, the geometry of a simplicial complex is only revealed when considering the different ways of orienting edges, triangles, tetrahedrons, etc., and determining how all the different oriented objects relate to each other. Since there are always two orientations of an edge, triangle, etc., it is unsurprising that there are certain "dualities" embedded in the geometry of simplicial complexes. For instance, in general the combinatorial Laplacian decomposes into two complementary "halves", the exception being the graph Laplacian (or 0-order Laplacian) and the highest-order Laplacian (or m-order Laplacian for a simplicial complex of dimension m). Therefore, it is possible that when generalizing spectral graph theory to higher dimensions, only one "half" of the spectral theory of combinatorial Laplacians will be revealed. Much of the work in this dissertation is spent overcoming precisely this kind of one-sided generalization to give a more comprehensive picture of the spectral theory of simplicial complexes.

The dualities that appear in higher dimensions on simplicial complexes are most intuitively understood by considering the analogous case of continuous manifolds. For manifolds with boundary, Hodge theory breaks down and the Laplacian must be restricted to a subspace of forms in order for its kernel to reflect the cohomology of the space. According to the famous Lefschetz duality theorem, there are two different subspaces to choose from: the subspace of Neumann forms and the subspace of Dirichlet forms. As their name implies, Neumann forms satisfy a sort of “Neumann” boundary condition while Dirichlet forms satisfy a “Dirichlet” boundary condition. If the Laplacian is restricted to Neumann forms, the resulting kernel is then isomorphic to absolute cohomology, and if it is restricted to Dirichlet forms, the resulting kernel is isomorphic to relative cohomology (i.e., cohomology relative to the boundary of the manifold). Spectral graph theory, in its most straightforward formulation, has always been intuitively identified with Neumann geometry. In this way, attempts to generalize spectral graph theory to higher dimensions have initially resulted in an exclusively Neumann perspective on the geometry of simplicial complexes. Perhaps the single biggest theoretical contribution this dissertation makes to the literature is it provides the missing Dirichlet perspective on the geometry of simplicial complexes.

2

History

In this section, we provide some historical context for the research presented in Chapters 3 and 4. We do not yet delve into the most directly relevant literature; that will be referred to as needed in Chapters 3 and 4.

2.1 Cheeger Numbers

Cheeger numbers were first defined on manifolds by Jeff Cheeger (after whom Cheeger numbers are named) in Cheeger (1970). Originally, only one Cheeger number was defined for manifolds, with its definition independent on whether the manifold had a boundary. Further work by Buser in **?** showed that in fact two separate definitions can be made, each independent of whether the manifold has a boundary. These are as follows:

Definition 1. *Let M be a d-dimensional Riemannian manifold, with or without boundary. Set*

$$h_N := \inf_{\emptyset \subsetneq S \subsetneq M} \frac{A(\partial S \setminus \partial M)}{\min\{\, Vol(S),\, Vol(M \setminus S)\}} \qquad \textit{and} \qquad h_D := \inf_{\emptyset \subsetneq S \subseteq M} \frac{A(\partial S)}{Vol(S)}$$

where the infimums are taken over all d-dimensional submanifolds S, $A()$ denotes $(d-1)$-dimensional area, Vol() denotes d-dimensional volume, and $A(\partial S \setminus \partial M)$ denotes the area of the part of the boundary of S that doesn't overlap with the boundary of M.

This is not the only way of defining Cheeger numbers. If one wants the infimums to be attained, for instance, one can replace submanifolds in the definition with $\mathbb{Z}_2$-valued currents (as studied in geometric measure theory).

What Cheeger and Buser proved is that

$$\lambda_* \geq \frac{1}{4} h_*^2$$

for $* = N, D$ where λ_N is the second smallest eigenvalue of the Laplace-Beltrami operator restricted to functions satisfying Neumann boundary conditions and λ_D is the smallest eigenvalue of the Laplace-Beltrami operator restricted to functions satisfying Dirichlet boundary conditions. This inequality is called Cheeger's inequality. Note that M is disconnected if and only if $h_N = \lambda_N = 0$, and similarly $h_D = \lambda_D = 0$ if and only if some connected component M_1 of M has no boundary and is orientable.

Starting with Cheeger's original paper, researchers have wondered if there is some higher-order analogue of Cheeger's inequality, that is, an inequality relating the spectral gap of a higher-order Hodge Laplacian to a Cheeger number. The underlying challenge here is that in general the spectral gaps of higher-order Hodge Laplacians can vary independently of each other (see Guerini and Savo (2003)). Thus, Cheeger number as originally defined can only reasonably relate to one spectral gap at a time. If a Cheeger inequality is to be found for higher-order Laplacians, it stands to reason that new higher-dimensional Cheeger numbers need to be defined.

2.2 Random Walks

The "usual" p-lazy random walk on a finite graph $G = (V, E)$, as described in Chung (1997), is a time-homogeneous Markov chain on V defined as follows. The walker starts at an initial vertex v_0. At each step of the walk, either the walker stays on the whatever vertex v it is on with probability p or with probability $1 - p$ jumps from the vertex v to one of the neighboring vertices (those sharing an edge with v), chosen at random with equal probability. We denote $\nu_n(v)$ as the marginal probability of the walker occupying vertex v after n steps of the walk, where $\nu_0(v_0) = 1$. The sequence of marginal probability distributions $\nu_0, \nu_1, \nu_2, \ldots$ evolves according to the probability transition matrix:

$$\nu_k = P\nu_{k-1}$$

where

$$P\nu(v') = p\nu(v') + \sum_{v' \sim v} \frac{1-p}{d_v}\nu(v).$$

Here $v' \sim v$ denotes that vertices v' and v share an edge, and d_v is the number of edges connected to v, known as the degree of vertex v. Alternatively, P can be written down as a matrix, with rows and columns indexed by V, such that

$$P_{v,v'} = \text{Prob}(v \to v') = \begin{cases} p & \text{if } v' = v \\ \frac{1-p}{d_v} & \text{if } v' \sim v \\ 0 & \text{else} \end{cases}.$$

A fundamental property of P is that it relates directly to the graph Laplacian L of G:

$$P = I - (1-p)LD^{-1}$$

where D is the degree matrix, i.e., the diagonal matrix with rows and columns indexed by V such that $D_{v,v} = d_v$. For our purposes, if $d_v = 0$ then $(D^{-1})_{v,v} = 1$. In probability theory, the probability transition matrix often used is P^t, the transpose of

P, such that $\nu_k = \nu_{k-1}P$. However, when relating probability transition matrices to Laplacians it is more convenient to use so-called left stochastic matrices as opposed to right stochastic matrices.

As a result of the above equation, P (and, hence, the random walk described above) can be studied purely in terms of properties of the graph Laplacian L. For instance, it is a basic fact that the spectrum of LD^{-1} is contained in the interval $[0, 2]$. The lower bound of 0 is always contained in the spectrum and 2 is an eigenvalue if and only if the graph is bipartite. Thus, the spectrum of P is contained in $[1-2(1-p), 1]$, 1 is always an eigenvalue of P, and -1 is an eigenvalue if and only if $p = 0$ and the graph contains a bipartite connected component. This implies the existence of a stationary distribution of the random walk, and implies that $\nu_n = P^n \nu_0$ converges to a stationary distribution for any initial distribution ν_0 if and only if $p > 0$ or $p = 0$ and the graph does not contain any bipartite connected component. If $p = 0$ and the graph contains a bipartite connected component, then all of the vertices in the bipartite connected component are 2-periodic states for the Markov chain. Furthermore, if the random walk starts at an initial vertex v_0 (so $\nu_0(v_0) = 1$) and ν_n converges, then G is connected (and the Markov chain is irreducible) if and only if $\lim_{n\to\infty} \nu_n(v) > 0$ for all $v \in V$. Finally, it can be proved that if $p \geq \frac{1}{2}$, then

$$\|\nu_k - \lim_{n\to\infty} \nu_n\|_2 = O([1-(1-p)\lambda]^k)$$

where $\|\cdot\|_2$ is the Euclidean norm and λ is the smallest nonzero eigenvalue of LD^{-1}.

These results, and other like them, have had many applications in theoretical and applied mathematics. As a result, it has long been an open question whether higher-dimensional random walks can be defined on simplicial complexes which have a similar relationship with the spectral theory of higher-order Laplacians. The difficulty, historically, has not been simply in defining random walks on higher-dimensional simplexes (edges, triangle, etc.), but in drawing any connection with higher-order

Laplacians. Unlike the graph Laplacian, when higher-order Laplacians are written down as matrices (in the "usual" way), they contain both positive and negative off-diagonal entries. This means that for any probability transition matrix P, an equation like $P = I - (1-p)LD^{-1}$ can never hold, since P by definition has only positive entries. Thus, new methods are needed to bridge the gap between the stochastic and spectral theories of simplicial complexes.

3

Isoperimetric Methods in the Spectral Theory of Simplicial Complexes

3.1 Introduction

3.1.1 Background

The Cheeger inequality (see Cheeger (1970); Buser (1980)) is a classic result that relates the isoperimetric constant of a manifold (with or without boundary) to the spectral gap of the Laplace-Beltrami operator. An analog of the manifold result was also found to hold on graphs (see Alon and Milman (1985); Alon (1986); Mohar (1989)) and is a prominent result in spectral graph theory. Given a graph G with vertex set V, the Cheeger number is the following isoperimetric constant

$$h := \min_{\emptyset \subsetneq S \subsetneq V} \frac{|\delta S|}{\min\{|S|, |\overline{S}|\}}$$

where δS is the set of edges connecting a vertex in S with a vertex in $\overline{S} = V \setminus S$. The Cheeger inequality on the graph relates the Cheeger number h to the algebraic connectivity λ (see Fiedler (1973)) which is the the second eigenvalue of the graph

Laplacian. It states that

$$2h \geq \lambda \geq \frac{h^2}{2\max_{v\in V} d_v}$$

where d_v is the number of edges connected to vertex v (also called the degree of the vertex). For more background on the Cheeger inequality see Chung (1997).

A key motivation for studying the Cheeger inequality has been understanding graph expansion in the sense of Hoory et al. (2006). A family of regular graphs of increasing size is said to be expanding if the algebraic connectivity λ of the graphs stay bounded away from 0, which, by way of the Cheeger inequality, is equivalent to saying that the Cheeger numbers h of the graphs stay bounded away from 0. Thus, h is used to study the expansion properties of graphs. A generalization of the Cheeger number to higher dimensions on simplicial complexes, based on ideas in Linial and Meshulam (2006); Meshulam and Wallach (2009), was defined and expansion properties studied in Dotterrer and Kahle (2012) via cochain complexes. In addition, it has been known since Eckmann (see Eckmann (1945)) that the graph Laplacian generalizes to higher dimensions on simplicial complexes. In particular one can generalize the notion of algebraic connectivity to higher dimensions using the cochain complex and relate an eigenvalue of the k-dimensional Laplacian to the k-dimensional Cheeger number. This raises the question of whether the Cheeger inequality has a higher-dimensional analog.

3.1.2 Main Results

In this chapter we examine the combinatorial Laplacian which is derived from a chain complex and a cochain complex. Precise definitions of the object studied and the results are given in section 2. We first state our negative result: for the cochain complex a natural Cheeger inequality does not hold. For an m-dimensional simplicial complex we denote λ^{m-1} as the analog of the spectral gap for dimension $m-1$ on the cochain complex and we denote h^{m-1} as the $(m-1)$-dimensional coboundary Cheeger number. In addition, let S_k be the set of k-dimensional simplexes and for any $s \in S_k$ let d_s be the number of $(k+1)$-simplexes incident to s. The following result is an informal statement of Proposition 12 and implies that there exists no Cheeger inequality of the following form for the cochain complex. Specifically, there are no constants p_1, p_2, C such that either of the inequalities

$$C(h^{m-1})^{p_1} \geq \lambda^{m-1} \qquad \text{or} \qquad \lambda^{m-1} \geq \frac{C(h^{m-1})^{p_2}}{\max_{s \in S_{m-1}} d_s}$$

holds in general for an m-dimensional simplicial complex X with $m > 1$. The case of $m = 1$ is the graph case, in which these inequalities hold with $p_1 = 1$ and $p_2 = 2$ (which is the classical Cheeger inequality on the graph).

For the chain complex we obtain a positive result: there is a direct analogue for the Cheeger inequality in certain well-behaved cases. Whereas the cochain complex is defined using the coboundary map, the chain complex is defined using the boundary map. Denote γ_m as the analog of the spectral gap for dimension m on the chain complex and h_m as the m-dimensional Cheeger number defined using the boundary map. If the m-dimensional simplicial complex X is an orientable pseudomanifold or satisfies certain more general conditions, then

$$h_m \geq \gamma_m \geq \frac{h_m^2}{2(m+1)}.$$

This inequality can be considered a discrete analog of the Cheeger inequality for manifolds with Dirichlet boundary condition (see Cheeger (1970); Buser (1980)).

3.1.3 Related Work

A probabilistic argument was used by Gundert and Wagner in Gundert and Wagner (2012) to show on the cochain complex there exists infinitely many simplicial complexes with $h^{m-1} = 0$ and $\lambda^{m-1} > c$ for some fixed constant $c > 0$, implying that one side of the Cheeger inequality cannot hold in general. However, this construction requires the complexes to have torsion in their integral homology groups due to the way h^{m-1} and λ^{m-1} relate to cohomology. In this chapter we show that even for torsion-free simplicial complexes there exist counterexamples that rule out both sides of a Cheeger inequality.

The analysis of the chain complex in this chapter is related to a paper by Fan Chung, Chung (2007), which introduces a notion of a Cheeger number on graphs with the analog of a Dirichlet boundary condition. We provide a detailed comparison in section 3.3.

Finally, it should be mentioned that the authors in Parzanchevski et al. (2012) prove a one-sided Cheeger-type inequality for λ^{m-1} using a modified higher-dimensional Cheeger number. The modified Cheeger number used is nonzero, and the inequality fails, unless the simplicial complex has complete skeleton.

3.2 Main Results

3.2.1 Simplicial Complexes

Since the concept of a Cheeger inequality is strongly associated to manifolds we focus in this chapter on abstract simplicial complexes that are analogous to well-behaved manifolds. In particular, we will focus on simplicial complexes that have geometric realizations homeomorphic to a Euclidean ball $B^m := \{x \in \mathbb{R}^m : \|x\|_2 \leq 1\}$. We will call such complexes simplicial m-balls

By a simplicial complex we mean an abstract finite simplicial complex. Simplicial complexes generalize the notion of a graph to higher dimensions. Given a set of vertices V, any nonempty subset $\sigma \subseteq V$ of the form $\sigma = \{v_0, v_1, \ldots, v_k\}$ is called a k-dimensional simplex, or k-simplex. A simplicial complex X is a finite collection of simplexes of various dimensions such that X is closed under inclusion, i.e., $\tau \subseteq \sigma$ and $\sigma \in X$ implies $\tau \in X$.

Given a simplicial complex X denote the set of k-simplexes of X as $S_k := S_k(X)$. We call X a simplicial m-complex if $S_m(X) \neq \emptyset$ but $S_{m+1}(X) = \emptyset$. Given two simplexes $\sigma \in S_k$ and $\tau \in S_{k+1}$ such that $\sigma \subset \tau$, we call σ a face of τ and τ a coface of σ. Two k-simplexes are lower adjacent if they share a common face and are upper adjacent if they share a common coface.

Every simplicial complex X has associated with it a geometric realization denoted $|X|$. The simplicial m-complex Σ^m consisting of a single m-simplex and its subsets has geometric realization homeomorphic to B^m. Thus, Σ^m is an example of a simplicial m-ball. A subdivision of a simplicial complex X is a simplicial complex X' such that $|X'| = |X|$ and every simplex of X' is, in the geometric realization, contained in a simplex of X. Thus, any subdivision of Σ^m is also a simplicial m-ball.

There is another convenient set of criteria under which a simplicial complex is a simplicial m-ball. A simplicial m-complex X is constructible if either (1) $X = \Sigma^m$ or (2) X can be decomposed into the union of two constructible simplicial m-subcomplexes $X = X_1 \cup X_2$ such that $X_1 \cap X_2$ is a constructible simplicial $(m-1)$-complex. If every $s \in S_{m-1}$ has at most two cofaces then X is said to be non-branching. In this case, every $s \in S_{m-1}$ with exactly one coface is called a boundary face of X. It is known (see Björner (1995)) that a the geometric realization of a non-branching constructible simplicial m-complex X is homeomorphic to B^m if X has at least one boundary face (otherwise it is homeomorphic to the sphere).

3.2.2 Chain and Cochain Complexes

Given a simplicial complex X and any field F, we can define the chain and cochain complexes of X over F. In this chapter we consider the fields $\mathbb{Z}_2$ and $\mathbb{R}$. Given a simplex $\sigma = \{v_0, v_1, \ldots, v_k\}$, σ can be ordered as a set. An orientation, denoted by $[v_0, v_1, \ldots, v_k]$ is an equivalence class of all even permutations of the given ordering. There are always two orientations for $k > 0$. The space of k-chains $C_k(F) :=$

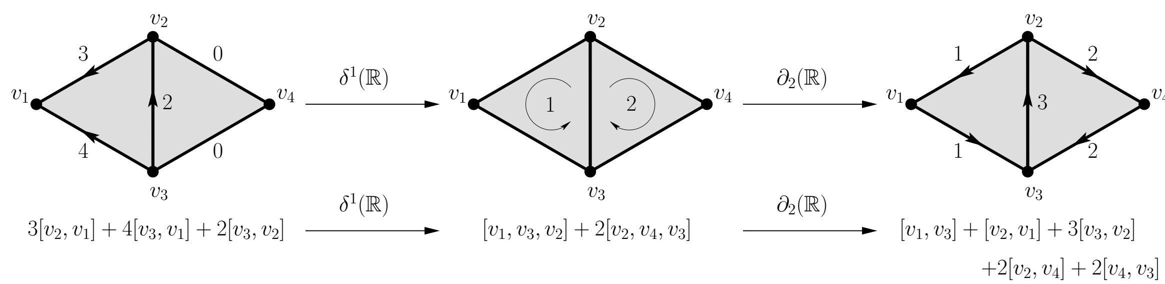

FIGURE 3.1: An example of $\partial(\mathbb{R})$ and $\delta(\mathbb{R})$.

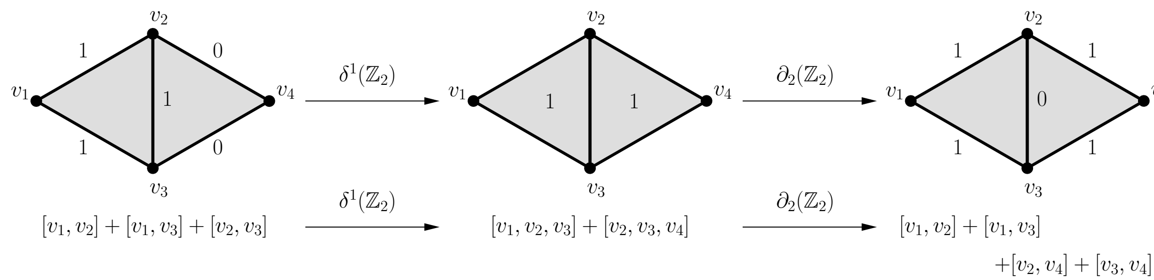

FIGURE 3.2: An example of $\partial(\mathbb{Z}_2)$ and $\delta(\mathbb{Z}_2)$.

$C_k(X;F)$ is the vector space of linear combinations of oriented k-simplexes with coefficients in F, with the stipulation that the two orientations of a simplex are negatives of each other in $C_k(F)$. The space of k-cochains $C^k(F) := C^k(X;F)$ is then defined to be the vector space dual to $C_k(F)$. These spaces are isomorphic and we will make no distinction between them. The boundary map $\partial_k(F) : C_k(F) \to C_{k-1}(F)$ is defined on the basis elements $[v_0, \dots, v_k]$ as

$$\partial_k[v_0, \dots, v_k] = \sum_{i=0}^{k} (-1)^i [v_0, \dots, v_{i-1}, v_{i+1}, \dots, v_k]$$

The coboundary map $\delta^{k-1}(F) : C^{k-1}(F) \to C^k(F)$ is then defined to be the transpose of the boundary map. When there is no confusion, we will denote the boundary and coboundary maps by ∂ and δ. It is easy to see that $\partial\partial = \delta\delta = 0$, so that $(C_k(F), \partial_k)$ and $(C^k(F), \delta^k)$ form chain and cochain complexes. See Figures 3.1 and 3.2 for examples of ∂ and δ on real and $\mathbb{Z}_2$ chains/cochains.

When $F = \mathbb{Z}_2$, positive and negative have no meaning and therefore no distinction is made between different orientations. In particular, it is possible to identify $C_k(\mathbb{Z}_2)$ and $C^k(\mathbb{Z}_2)$ with S_k as sets. Throughout this chapter, we will identify a k-chain/k-cochain ϕ over $\mathbb{Z}_2$ with the subset $\phi \subset S_k$ of k-simplexes to which ϕ assigns the coefficient 1 (as opposed to 0).

The homology and cohomology vector spaces of X over F are

$$H_k(F) := H_k(X;F) = \frac{\ker \partial_k}{\operatorname{im} \partial_{k+1}} \quad \text{and} \quad H^k(F) := H^k(X;F) = \frac{\ker \delta^k}{\operatorname{im} \delta^{k-1}}.$$

It is known from the universal coefficient theorem that $H^k(F)$ is the vector space dual to $H_k(F)$.

3.2.3 Laplacians and Eigenvalues

The k-th Laplacian of X is defined to be

$$L_k := L_k^{\text{up}} + L_k^{\text{down}}$$

where

$$L_k^{\text{up}} = \partial_{k+1}(\mathbb{R})\delta^k(\mathbb{R}) \quad \text{and} \quad L_k^{\text{down}} = \delta^{k-1}(\mathbb{R})\partial_k(\mathbb{R}).$$

By way of Rayleigh quotients, the smallest nontrivial eigenvalue of L_k^{up} and L_k^{down} are given by

$$\lambda^k = \min_{\substack{f \in C^k(\mathbb{R}) \\ f \perp \operatorname{im} \delta}} \frac{\|\delta f\|_2^2}{\|f\|_2^2} = \min_{\substack{f \in C^k(\mathbb{R}) \\ f \notin \operatorname{im} \delta}} \frac{\|\delta f\|_2^2}{\min_{g \in \operatorname{im} \delta} \|f+g\|_2^2},$$

$$\lambda_k = \min_{\substack{f \in C_k(\mathbb{R}) \\ f \perp \operatorname{im} \partial}} \frac{\|\partial f\|_2^2}{\|f\|_2^2} = \min_{\substack{f \in C_k(\mathbb{R}) \\ f \notin \operatorname{im} \partial}} \frac{\|\partial f\|_2^2}{\min_{g \in \operatorname{im} \partial} \|f+g\|_2^2},$$

where $\|\cdot\|_2$ denotes the Euclidean norm on both $C^k(\mathbb{R})$ and $C_k(\mathbb{R})$. It is well known that the nonzero spectrum of L_k is the union of the nonzero spectrum of L_k^{up} with the nonzero spectrum of L_k^{down}. Thus, the smallest nonzero eigenvalue of L_k is either λ^k or λ_k assuming one of them is nonzero. In addition, the nonzero spectrum of L_k^{up}

is the same as the nonzero spectrum of L_{k+1}^{down}. Thus, $\lambda^k = \lambda_{k+1}$ whenever λ^k, λ_{k+1} are both nonzero.

The relationship between eigenvalues and homology/cohomology is as follows:

$$\begin{array}{ccc} \lambda_k = 0 & & \lambda^k = 0 \\ \Updownarrow & \text{and} & \Updownarrow \\ H_k(\mathbb{R}) \neq 0 & & H^k(\mathbb{R}) \neq 0. \end{array}$$

If we pass to the reduced cochain complex, λ^0 becomes the algebraic connectivity (or Fiedler number) of a graph (see Fiedler (1973)) and $\lambda^0 = 0 \Leftrightarrow \widetilde{H}^0(\mathbb{R}) \neq 0$.

3.2.4 Cheeger Numbers

Higher-dimensional Cheeger numbers were first stated in Dotterrer and Kahle (2012) to capture a higher-dimensional notion of expanders. They are defined via the coboundary map as follows:

Definition 2. *Let $\|\cdot\|$ denote the Hamming norm on $C^k(\mathbb{Z}_2)$. The k-th (coboundary) Cheeger number of X is*

$$h^k := \min_{\substack{\phi \in C^k(\mathbb{Z}_2) \\ \phi \notin \text{im } \delta}} \frac{\|\delta\phi\|}{\min_{\psi \in \text{im } \delta} \|\phi + \psi\|}.$$

A similar definition can be given for the boundary map.

Definition 3. *Let $\|\cdot\|$ also denote the Hamming norm on $C_k(\mathbb{Z}_2)$. The k-th boundary Cheeger number of X is*

$$h_k := \min_{\substack{\phi \in C_k(\mathbb{Z}_2) \\ \phi \notin \text{im } \partial}} \frac{\|\partial\phi\|}{\min_{\psi \in \text{im } \partial} \|\phi + \psi\|}.$$

The relationship between Cheeger numbers and homology/cohomology is as follows:

$$\begin{array}{ccc} h_k = 0 & & h^k = 0 \\ \Updownarrow & \text{and} & \Updownarrow \\ H_k(\mathbb{Z}_2) \neq 0 & & H^k(\mathbb{Z}_2) \neq 0 \ . \end{array}$$

If we pass to the reduced cochain complex, h^0 becomes the Cheeger number of a graph (see Dotterrer and Kahle (2012)) and $h^0 = 0 \Leftrightarrow \widetilde{H}^0(\mathbb{Z}_2) \neq 0$.

Often, we speak of a cochain that attains the minimum in the definition of the Cheeger number (in the graph case these are Cheeger cuts). We will say that $\phi \in C^k(\mathbb{Z}_2)$ attains h^k if $h^k = \frac{\|\delta\phi\|}{\|\phi\|}$. The same terminology will be used for h_k.

3.2.5 Additional Notation and Preliminary Results

Here we collect some interesting results concerning Cheeger numbers which will be needed later in section 3.2.6. Lemma 4 says that h_1 has a very simple interpretation in terms of the diameter of the simplicial complex. Lemma 6 says that h^{m-1} also has a very simple interpretation in terms of the radius.

We define the diameter of a simplicial m-complex X as follows. Given two vertices $v_1, v_2 \in S_0$, we define the distance between them to be the quantity

$$\mathrm{dist}(v_1, v_2) := \min\{\|\phi\| : \phi \in C_1(\mathbb{Z}_2) \text{ and } \partial\phi = v_1 + v_2\}$$

Any chain ϕ attaining the minimum is called a geodesic. Note that for any geodesic ϕ, $h_1 \leq \frac{2}{\|\phi\|}$. For our purposes, $\mathrm{dist}(v_1, v_2) = 0$ if v_1, v_2 are not in the same connected component. The diameter of X is then defined to be

$$\mathrm{diam}(X) := \max_{v_1, v_2 \in S_0} \mathrm{dist}(v_1, v_2).$$

As it turns out, h_1 is strongly related to the diameter of a simplicial complex.

Lemma 4. *Given a simplicial m-complex X with $m \geq 1$ and satisfying $H_1(\mathbb{Z}_2) = 0$, h_1 is attained by a geodesic and hence*

$$h_1 = \frac{2}{\mathrm{diam}(X)}$$

.

Proof. Suppose that $\phi \in C_1(\mathbb{Z}_2)$ attains h_1. Clearly, $\|\partial\phi\|$ must be even and nonzero. What we will show is that we can assume $\|\partial\phi\| = 2$. Thinking of ϕ as a graph (consisting of the edges in ϕ and their vertices), it is also clear that every connected component ϕ_i of ϕ has $\|\partial\phi_i\|$ even. For every pair of vertices in $\partial\phi_i$, there exists a geodesic in X with the given pair of vertices as its boundary. Thus, there exist geodesics $\psi_1, \ldots, \psi_q$ such that $\partial\psi_j$ is a distinct pair of vertices in $\partial\phi$ for all j and $\partial(\psi_1 + \cdots + \psi_q) = \partial\phi$. Since ϕ attains h_1 and $H_1(\mathbb{Z}_2) = 0$,

$$\|\phi\| = \min_{\psi \in \operatorname{im}\partial} \|\phi + \psi\| = \min_{\partial\psi = \partial\phi} \|\psi\|$$

In other words, ϕ is a 1-chain of smallest norm with boundary $\partial\phi$. Thus, $\|\psi_1 + \cdots + \psi_q\| \geq \|\phi\|$. Now,

$$\begin{aligned} h_1 &= \frac{\|\partial\phi\|}{\|\phi\|} \\ &\geq \frac{\|\partial(\psi_1 + \cdots + \psi_q)\|}{\|\psi_1 + \cdots + \psi_q\|} \\ &\geq \frac{2 + \cdots + 2}{\|\psi_1\| + \cdots + \|\psi_q\|} \\ &\geq \min\left\{\frac{2}{\|\psi_1\|}, \ldots, \frac{2}{\|\psi_q\|}\right\} \\ &\geq h_1 \end{aligned}$$

and therefore $h_1 = \min\left\{\frac{2}{\|\psi_1\|}, \ldots, \frac{2}{\|\psi_q\|}\right\}$. Here we are using the general inequality $\frac{a_1+a_2+\cdots+a_k}{b_1+b_2+\cdots+b_k} \geq \min_i \frac{a_i}{b_i}$, valid for all $a_1, \ldots, a_k, b_1, \ldots, b_k > 0$. Hence, $h_1 = \frac{2}{\|\psi_j\|}$ for some geodesic ψ_j. This completes the proof. $\square$

While the diameter is defined in terms of 1-chains, we define the radius in terms of $(m-1)$-cochains as follows. Given a simplicial m-complex X, we define the depth of an m-simplex σ to be

$$\operatorname{depth}(\sigma) := \min\{\|\phi\| : \phi \in C^{m-1}(\mathbb{Z}_2), \delta\phi = \sigma\}.$$

Any minimizing ϕ will be said to be a depth-attaining cochain for σ. Note that for any such ϕ, $h^{m-1} \leq \frac{1}{\|\phi\|}$. All m-simplexes have a defined depth when $H_m(\mathbb{Z}_2)$ is trivial. In this case, we define the radius of X to be

$$\operatorname{rad}(X) := \max_{\sigma \in S_m} \operatorname{depth}(\sigma).$$

Depth-attaining cochains have a very predictable structure for non-branching simplicial complexes, a fact which we will use later in proving Proposition 12. Roughly speaking, Lemma 5 says that if ϕ is depth-attaining for σ, then ϕ is a linear non-intersecting sequence of $(m-1)$-simplexes starting with a face of σ and ending with a boundary face. For the statement and proof of this Lemma we define the star $\operatorname{st}(s)$ of a simplex s to be the set of cofaces of s.

Lemma 5. *Let X be a simplicial m-complex such that every $s \in S_{m-1}$ has at most two cofaces. Suppose that $\sigma \in S_m$ has depth d and ϕ is a depth-attaining cochain for σ. Then there is a sequence $s_1, s_2, \ldots, s_d$ of distinct $(m-1)$-simplexes and a sequence $\sigma = \sigma_1, \sigma_2, \ldots, \sigma_d$ of distinct m-simplexes satisfying*

1. $\phi = \sum_{i=1}^{d} s_i$,
2. $\operatorname{st}(s_i) = \{\sigma_i, \sigma_{i+1}\}$ *for* $i < d$,
3. $\operatorname{st}(s_d) = \{\sigma_d\}$.

Proof. Assume $\phi = \sum_{i=1}^{d} s_i$. Clearly, at least one of the s_i must have σ as a coface, so WLOG we can assume s_1 has $\sigma = \sigma_1$ as a coface. If s_1 is a boundary face, we are done and $d = 1$. If not, then s_1 has another coface σ_2. In this case, if there are no other s_i with σ_2 as a coface then we arrive at the contradiction that $\delta\phi$ contains

σ_2, i.e., $\delta\phi \neq \sigma$. Thus, there is another s_i with σ_2 as a coface, which we can assume WLOG is s_2.

We proceed by induction. Suppose that for $k > 1$ there is a sequence $\sigma_1, \sigma_2, \ldots, \sigma_k$ of distinct m-simplexes such that $\delta(s_1 + \cdots + s_{k-1}) = \sigma + \sigma_k$ where $\mathrm{st}(s_i) = \{\sigma_i, \sigma_{i+1}\}$ for all i. Then we can find another s_i, $i > k$, which we can assume WLOG is s_k and which has σ_k as a coface. If no such s_i exists then $\delta\phi \neq \sigma$. If s_k is a boundary face we are done and $d = k$. If s_k has σ_{k+1} as a second coface and $\sigma_{k+1} = \sigma_i$ for some $i < k$ then $s_i + \ldots + s_k$ is a cocycle, but this means that $\delta(\phi - s_i - \cdots - s_k) = \sigma$ so ϕ is not depth-attaining. Otherwise, $\sigma_1, \sigma_2, \ldots, \sigma_{k+1}$ is a sequence of distinct m-simplexes such that $\delta(s_1 + \cdots + s_k) = \sigma + \sigma_{k+1}$ where $\mathrm{st}(s_i) = \{\sigma_i, \sigma_{i+1}\}$ for all i. This leaves us back where we started. By induction, we can continue this process until $k = d$ and s_d is a boundary face. □

Lemma 6. *Let X be a simplicial m-complex with $H^{m-1}(\mathbb{Z}_2) = 0$ and $H_m(\mathbb{Z}_2) = 0$. Then h^{m-1} is attained by a depth-attaining cochain and hence*

$$h^{m-1} = \frac{1}{\mathrm{rad}(X)}.$$

Proof. Suppose ψ attains h^{m-1} and $\delta\psi$ is a sum of distinct m-simplexes $\sigma_1, \ldots, \sigma_q$ with depth-attaining cochains $\psi_1, \ldots, \psi_q$. Clearly $\|\psi\| \leq \|\psi_1\| + \cdots + \|\psi_q\|$, so

$$\begin{aligned} h^{m-1} &= \frac{q}{\|\psi\|} \\ &\geq \frac{1 + \cdots + 1}{\|\psi_1\| + \cdots + \|\psi_q\|} \\ &\geq \min\left\{\frac{1}{\|\psi_1\|}, \ldots, \frac{1}{\|\psi_q\|}\right\} \\ &\geq h^{m-1} \end{aligned}$$

and therefore $h^{m-1} = \min\left\{\frac{1}{\|\psi_1\|}, \ldots, \frac{1}{\|\psi_q\|}\right\}$. Here we are using the general inequality

$\frac{a_1+a_2+\cdots+a_k}{b_1+b_2+\cdots+b_k} \geq \min_i \frac{a_i}{b_i}$, valid for all $a_1,\ldots,a_k,b_1,\ldots,b_k > 0$. Hence, $h^{m-1} = \frac{1}{\|\psi_j\|}$ for some depth-attainng cochain ψ_j. This completes the proof. □

An interesting result which will not be used in this chapter is a Cheeger-type inequality for the special case $X = \Sigma^m$.

Lemma 7. *Recall Σ^m is the simplicial complex induced by an m-simplex. The following holds for all k.*

1. $h^k(\Sigma^{m-1}) \geq \frac{m}{k+2}$
2. $h_k(\Sigma^{m-1}) \geq \frac{m}{m-k}$.

The reason this result is Cheeger-type is because all the Laplacian eigenvalues of all dimensions for Σ^{m-1} are equal to m (this is easily seen from the characterization of the Laplacian in Muhammad and Egerstedt (2006)). Part (1) of this Lemma was proved by Meshulam and Wallach in Meshulam and Wallach (2009) (who, even though they did not define the Cheeger number, still worked with its numerator and denominator separately). Their proof can be easily modified to prove part (2) of the Lemma.

3.2.6 Main Results

We now state the main results of this chapter – there exists a Cheeger-type inequality in the top dimension for the chain complex but not for the cochain complex.

To state the results we need the following notion of orientational similarity. Two oriented lower adjacent k-simplexes are dissimilarly oriented if they induce the same orientation on the common face. In other words, if $\sigma = [v_0,\ldots,v_k]$ and $\tau = [w_0,\ldots,w_k]$ share the face $\{u_0,\ldots,u_{k-1}\}$, then σ and τ are dissimilarly oriented if $\partial(\mathbb{R})\sigma$ and $\partial(\mathbb{R})\tau$ assign the same coefficient ($+1$ or -1) to the oriented simplex

$[u_0, \ldots, u_{k-1}]$. Otherwise, they are said to be similarly oriented. If X is a simplicial m-complex and all its m-simplices can be oriented similarly, then X is called orientable.

We first state the positive result – there is a Cheeger-type inequality for the chain complex.

Theorem 8. *Let X be a simplicial m-complex, $m > 0$.*

(1) Let $\phi \in C_m(\mathbb{Z}_2)$ minimize the quotient in

$$h_m := \min_{\substack{\phi \in C_m(\mathbb{Z}_2) \\ \phi \notin \mathrm{im}\, \partial}} \frac{\|\partial \phi\|}{\min_{\psi \in \mathrm{im}\, \partial} \|\phi + \psi\|}.$$

If all m-simplexes in ϕ can be similarly oriented, then $h_m \geq \lambda_m$.

(2) Assume that every $(m-1)$-dimensional simplex is incident to at most two m-simplexes. Then

$$\lambda_m \geq \frac{h_m^2}{2(m+1)}.$$

The first statement is the analog of the Buser inequality for graphs. The second statement is an analog of the Cheeger inequality for graphs, as well as the Cheeger inequality for a manifold with Dirichlet boundary conditions. The constraint that every $(m-1)$-simplex has at most two cofaces enforces the boundary condition. The hypotheses required for both inequalities are always satisfied by orientable pseudo-manifolds.

The hypotheses required by the Theorem cannot be removed, as proved by the following two examples.

Example 9 (Real Projective Plane)**.** *Given a triangulation X of $\mathbb{R}P^2$ (see Figure 3.3) we know that $H_2(\mathbb{Z}_2) \neq 0$ while $H_2(\mathbb{R}) = 0$, so that $h_2 = 0 \neq \lambda_2$. This is due* *to the nonorientability of $\mathbb{R}P^2$. The chain $\phi \in C_2(\mathbb{Z}_2)$ containing every m-simplex has no boundary. However, the m-simplexes cannot all be similarly oriented, so that there is no corresponding boundaryless chain in $C_2(\mathbb{R})$. As a result, the hypothesis used in part (1) of the Theorem cannot in general be removed.*

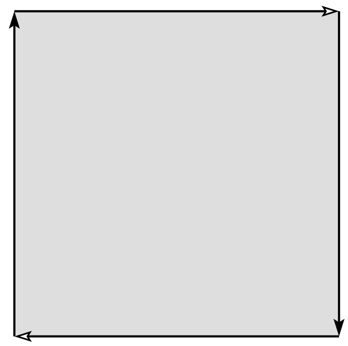

FIGURE 3.3: The fundamental polygon of $\mathbb{R}P^2$.

Example 10. *Let G_k be a graph with $2k$ vertices of degree one, half of which connect to one end of an edge and the other half connect to the other end (see figure 3.4). Clearly, $h^0(G_k) = \frac{1}{k+1}$ while Lemma 4 implies $h_1 = \frac{2}{3}$. By the Buser inequality for graphs, $\lambda^0 \le \frac{2}{k+1}$ and since $\lambda_1 = \lambda^0$, this means that $\lambda_1 \to 0$. As a result, we conclude that the hypothesis used in part (2) of the Theorem cannot be removed.*

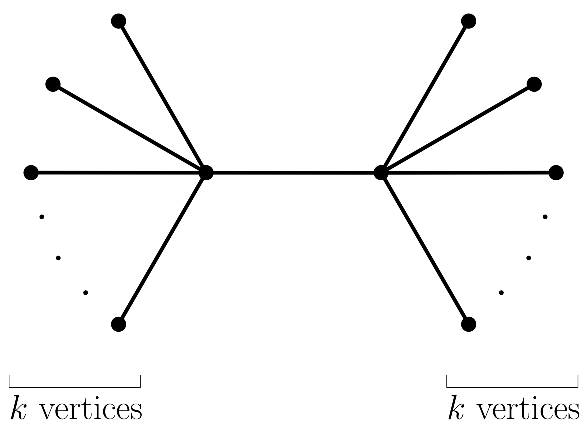

FIGURE 3.4: The family of graphs G_k.

The next example shows what it typically looks like when both inequalities hold.

Example 11. *Let Y be the simplicial 2-complex shown in figure 3.5. Clearly, Y satisfies the conditions for Theorem 8 to hold. It is not hard to compute in this case the $\mathbb{Z}_2$ 2-chain that minimizes h_2, and it is shown in 3.6. The corresponding eigenvector of λ_2 is depicted in figure 3.7.*

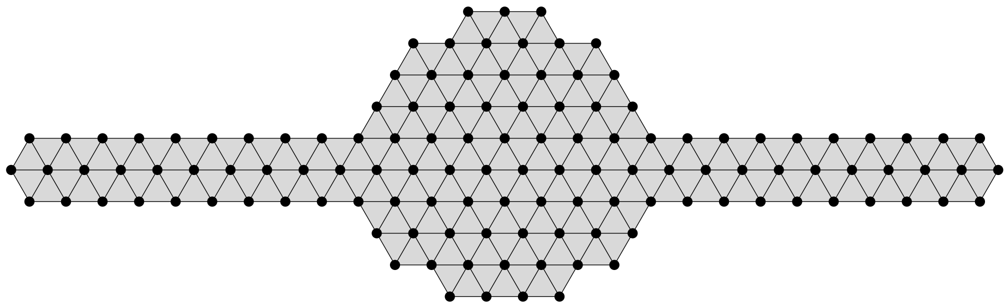

FIGURE 3.5: The simplicial 2-complex Y.

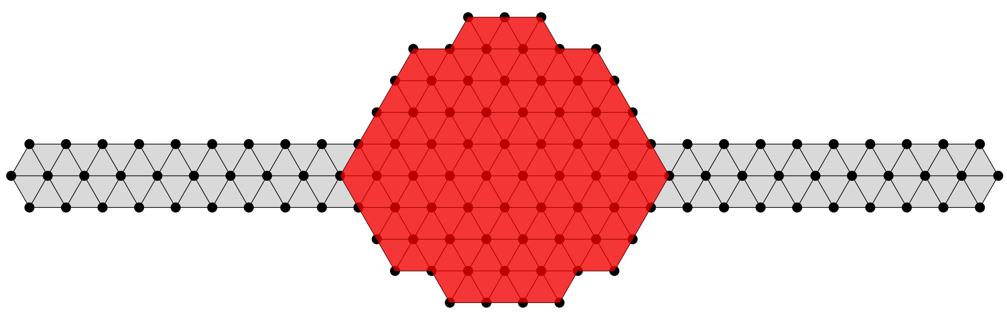

FIGURE 3.6: The chain $\phi \in C_2(\mathbb{Z}_2)$ that minimizes h_2. The chain assigns a 1 to all colored triangles and 0 to all else.

Proof of Theorem 8. Given the hypotheses, λ_m is a linear programming relaxation of h_m. Let $g \in C_m(\mathbb{R})$ be the chain which assigns a 1 to every simplex in ϕ (all of them similarly oriented) and a 0 to every other simplex. Then

$$h_m = \frac{\|\partial \phi\|}{\|\phi\|} = \frac{\|\partial g\|_2^2}{\|g\|_2^2} \geq \min_{\substack{f \in C_m(\mathbb{R}) \\ f \neq 0}} \frac{\|\partial f\|_2^2}{\|f\|_2^2} = \lambda_m.$$

$\square$

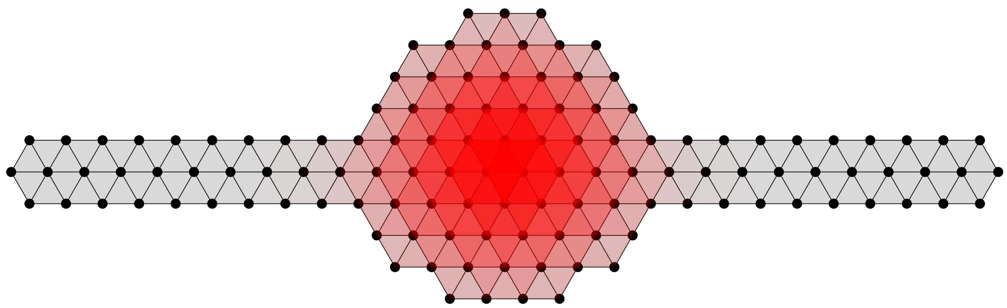

FIGURE 3.7: The eigenvector $f \in C_2(\mathbb{R})$ of λ_2. The function f assigns values near 1 for the central triangles and values near 0 to all other triangles.

Proof of Theorem 8. Let f be an eigenvector of λ_m and for any oriented m-simplex σ let $f(\sigma)$ denote the coefficient assigned to σ by f. Orient the m-simplexes of X so that all the values of f are non-negative and let $S_m^{\text{or}}(X)$ be the set of oriented m-simplices of X. We do not assume the m-simplexes are similarly oriented. Number the m-simplexes from 1 to $N := |S_m^{\text{or}}(X)|$ in increasing order of f:

$$0 \leq f(\sigma_1) \leq f(\sigma_2) \leq \cdots \leq f(\sigma_N).$$

To aid us in the proof, we introduce a new simplicial m-complex X' which contains X as a subcomplex and which is defined as follows: for every boundary face $s = \{v_0, \ldots, v_{m-1}\}$ in X create a new vertex v and a new m-simplex $\sigma = \{v_0, \ldots, v_{m-1}, v\}$ which includes v and s. These new m-simplexes will be called border facets. Give the border facets any orientation and let $F_m^{\text{or}}(X')$ be the set of oriented border facets. We can extend f to be a function on $S_m^{\text{or}}(X) \cup F_m^{\text{or}}(X')$ by defining $f(\sigma) = 0$ for any $\sigma \in F_m^{\text{or}}(X')$. Let $M := |F_m^{\text{or}}(X')|$ and number the oriented border facets in any order:

$$F_m^{\text{or}}(X') = \{\sigma_0, \sigma_{-1}, \ldots, \sigma_{1-M}\}.$$

The intuition behind introducing the border facets comes from the analogy with the continuous Cheeger inequality for functions satisfying Dirichlet boundary conditions (see Cheeger (1970)). In our case, the Dirichlet boundary condition is implicit

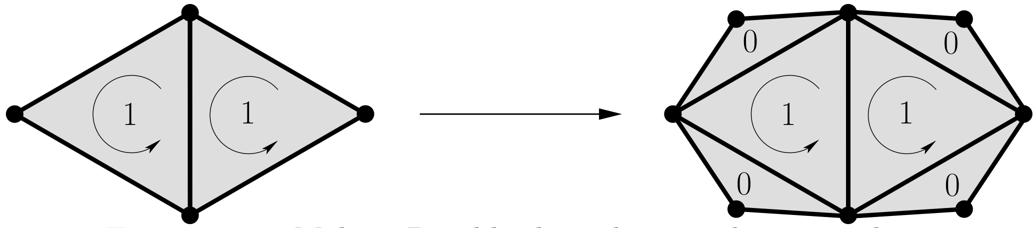

FIGURE 3.8: Making Dirichlet boundary conditions explicit.

in the fact that f is defined on m-simplexes (as opposed to vertices). The border facets represent the boundary of the m-dimensional part of X, and f is in fact zero on them. See Figure 3.8 for a depiction. In this analogy, h_m plays the part of the Cheeger number defined as in Cheeger (1970) for manifolds with boundary.

When two simplexes σ, τ are lower adjacent we write $\sigma \sim \tau$. Now define

$$C_i = \{\{\sigma_j, \sigma_k\} : 1 - M \leq j \leq i < k \leq N \text{ and } \sigma_j \sim \sigma_k\}$$

and

$$h[f] = \min_{0 \leq i \leq N-1} \frac{|C_i|}{N - i}.$$

Observe that $h[f] \geq h_m$.

We now finish the theorem. The following summations are taken over all oriented m-simplexes in $S_m^{\mathrm{or}}(X) \cup F_m^{\mathrm{or}}(X')$.

$$\lambda_m = \frac{\sum_{\sigma\sim\tau}(f(\sigma) \pm f(\tau))^2}{\sum_\sigma f(\sigma)^2}, \tag{1}$$

$$= \frac{\sum_{\sigma\sim\tau}(f(\sigma) \pm f(\tau))^2}{\sum_\sigma f(\sigma)^2} \cdot \frac{\sum_{\sigma\sim\tau}(f(\sigma) \mp f(\tau))^2}{\sum_{\sigma\sim\tau}(f(\sigma) \mp f(\tau))^2},$$

$$\geq \frac{\left(\sum_{\sigma\sim\tau} |f(\sigma)^2 - f(\tau)^2|\right)^2}{(\sum_\sigma f(\sigma)^2) \cdot (\sum_{\sigma\sim\tau}(f(\sigma) \mp f(t_2))^2)}, \tag{3}$$

$$\geq \frac{\left(\sum_{\sigma\sim\tau} |f(\sigma)^2 - f(\tau)^2|\right)^2}{(\sum_\sigma f(\sigma)^2) \cdot (2\sum_{\sigma\sim\tau} f(\sigma)^2 + f(\tau)^2)},$$

$$= \frac{\left(\sum_{\sigma\sim\tau} |f(\sigma)^2 - f(\tau)^2|\right)^2}{(\sum_\sigma f(\sigma)^2) \cdot 2(m+1) \cdot (\sum_\sigma f(\sigma)^2)},$$

$$= \frac{\left(\sum_{i=0}^{N-1}(f(\sigma_{i+1})^2 - f(\sigma_i)^2)|C_i|\right)^2}{2(m+1)\cdot\left(\sum_\sigma f(\sigma)^2\right)^2}, \tag{6}$$

$$\geq \frac{\left(\sum_{i=0}^{N-1}(f(\sigma_{i+1})^2 - f(\sigma_i)^2)h[f](N-i)\right)^2}{2(m+1)\cdot\left(\sum_\sigma f(\sigma)^2\right)^2},$$

$$= \frac{h[f]^2}{2(m+1)}\cdot\frac{\left(\sum_\sigma f(\sigma)^2\right)^2}{\left(\sum_\sigma f(\sigma)^2\right)^2},$$

$$\geq \frac{h_m^2}{2(m+1)}.$$

Step (1) follows from the Rayleigh quotient characterization of λ_m and step (3) follows from the Cauchy-Schwarz inequality. We prove the statement for step (6) below.

We want to show

$$\sum_{\sigma\sim\tau}|f(\sigma)^2 - f(\tau)^2| = \sum_{i=0}^{N-1}(f(\sigma_{i+1})^2 - f(\sigma_i)^2)|C_i|.$$

This can be seen by counting the number of times each $f(\sigma_i)^2$ appears in each sum. In the left hand sum, each $f(\sigma_i)^2$ appears a number of times equal to

$$\Delta_i := |\{\{\sigma_j,\sigma_i\} : j < i \text{ and } \sigma_j \sim \sigma_i\}| - |\{\{\sigma_i,\sigma_k\} : i < k \text{ and } \sigma_i \sim \sigma_k\}|.$$

On the other hand, each $f(\sigma_i)^2$ appears $|C_{i-1}| - |C_i|$ times in the right hand sum. To see that these are the same, note that for each pair $\{\sigma_j,\sigma_k\}$ in C_{i-1}, either $k = i$ or else $\{\sigma_j,\sigma_k\}$ is in C_i as well, meaning it is canceled in the difference. Similarly, for each pair $\{\sigma_j,\sigma_k\}$ in C_i, either $j = i$ or else $\{\sigma_j,\sigma_k\}$ is in C_{i-1} as well, again meaning it is canceled. Thus

$$|C_{i-1}| - |C_i| = \Delta_i.$$

This completes the proof. □

We now state the negative result – the analogous Cheeger-type inequality for the cochain complex does not hold.

Proposition 12. *For every $m > 1$, there exist families of simplicial m-balls X_k and Y_k such that*

(1) for X_k, $\lambda^{m-1}(X_k) \geq \frac{(m-1)^2}{2(m+1)}$ for all k but $h^{m-1}(X_k) \to 0$ as $k \to \infty$.

(2) for Y_k, $\lambda^{m-1}(Y_k) \leq \frac{1}{m^{k-1}}$ for $k > 1$ but $h^{m-1}(Y_k) \geq \frac{1}{k}$ for all k.

As mentioned in the introduction, it has already been shown in Gundert and Wagner (2012) that there exist infinite families of simplicial complexes for which $h^{m-1} = 0$ but λ^{m-1} is bounded away from 0. Such a construction relies on the presence of torsion in the integral homology groups. Indeed, any simplicial complex with torsion can be used to show that the inequality $(h^k)^p \geq C\lambda^k$ need not hold in general for any p,$C > 0$, and $k > 0$. A good example is $\mathbb{RP}^2$ which has $H^1(\mathbb{Z}_2) \neq 0$ and $H^2(\mathbb{Z}_2) \neq 0$ but $H^1(\mathbb{R}) = 0$ and $H^2(\mathbb{R}) = 0$. By contrast, the example presented here is a family of orientable simplicial complexes, proving that the failure of the Cheeger inequality to hold is not simply the result of torsion.

The fact that both families X_k and Y_k are simplicial m-balls helps show the degree to which the Cheeger inequality fails to hold even for 'nice' simplicial complexes.

The proof of Proposition 12 puts together much of what appears earlier in this chapter. To show that X_k is a simplicial m-ball we will need to prove that it is constructible and non-branching. The Y_k will be defined by subdividing Σ^m, implying that it too is a simplicial m-ball. To compute the values of h^{m-1} for X_k and Y_k we make use of Lemmas 5 and 6. Computing h_m will involve simple counting. By Theorem 8 and the fact that $\lambda_m = \lambda^{m-1}$, we can use our estimate of h_m to estimate λ^{m-1}, finishing the proof.

Now to begin the proof. We define the family X_k recursively. To begin with, we let X_1 be Σ^m, the simplicial complex induced by a single m-simplex. Note that

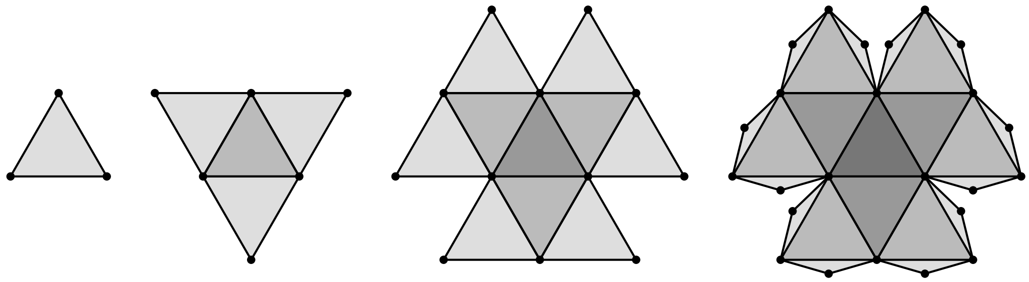

FIGURE 3.9: The first four iterations X_1, X_2, X_3, X_4 of X_k in dimension 2. The 2-simplexes have been shaded according to their depth.

$h_m(X_1) = m + 1$ and $h^{m-1}(X_1) = 1$. Then, given X_k, we define X_{k+1} by gluing m-simplexes on to X_k as follows: for each boundary face $s = \{v_0, \ldots, v_{m-1}\}$ in X_k we create a new vertex v and a new m-simplex $\sigma = \{v_0, \ldots, v_{m-1}, v\}$ which includes v and s. A picture of the first few iterations of X_k for the case $m = 2$ can be seen in Figure 3.9.

Clearly, X_1 is a simplicial m-ball. The following two lemmas prove that indeed every X_k is a simplicial m-ball.

Lemma 13. *X_k is constructible for all k.*

Proof. The proof is by induction. We know X_1 is constructible. Assuming that X_k is constructible, we must prove that X_{k+1} is constructible. This reduces to proving that gluing a single m-simplex to X_k along a boundary face preserves constructibility. Let X_k' be the result of taking a boundary face $s = \{v_0, \ldots, v_{m-1}\}$ in X_k and adding a new vertex v and a new m-simplex $\sigma = \{v_0, \ldots, v_{m-1}, v\}$ which includes v and s. Then X_k' can be decomposed as the union of X_k and the simplicial subcomplex $T = \Sigma^m$ consisting of σ and its subsets, both of which are constructible m-complexes. Furthermore, the intersection of X_k and T is Σ^{m-1}, which is constructible. Therefore, X_k' is constructible by definition. □

Lemma 14. *X_k is non-branching for all k.*

Proof. The proof is again by induction. We know that X_1 is non-branching. Assume this is true for X_k as well. By construction, $s \in S_{m-1}(X_k)$ has another coface in $S_m(X_{k+1})$ if and only if s has only one coface in $S_m(X_k)$. The new $(m-1)$-simplexes are the boundary faces of X_{k+1} and thus have exactly one coface. Thus, the total number of cofaces of every $(m-1)$-simplex in X_{k+1} is either one or two. $\square$

As mentioned in the introduction, constructible non-branching simplicial m-complexes are simplicial m-balls. Thus, every X_k is a simplicial m-ball.

To prove part (1) of Proposition 12, we need to keep track of how the Cheeger numbers $h^{m-1}(X_k)$ and $h_m(X_k)$ change with k. This is accomplished in the following two lemmas.

Lemma 15. *$h^{m-1}(X_k) = \frac{1}{k}$ for all k.*

Proof. By Lemma 6, $h^{m-1}(X_k) = \frac{1}{\operatorname{rad}(X_k)}$. For $k = 1$, $\operatorname{rad}(X_1) = 1$. Now suppose that $\operatorname{rad}(X_k) = k$. We will prove that in passing from X_k to X_{k+1}, all m-simplexes originally in X_k have their depth increased by exactly 1 (we already know the new m-simplexes in X_{k+1} have depth 1).

If $\tau \in S_m(X_k)$ has depth d and ϕ is a depth-attaining cochain for τ in X_k, then ϕ is a sum of a sequence $\{s_i\}_{i=1}^{d}$ of $(m-1)$-simplexes satisfying the conditions in Lemma 5. All of those conditions are preserved in going from X_k to X_{k+1}, except that s_d is no longer a boundary face. Instead, if $s_d = \{v_0, \ldots, v_{m-1}\}$ then a new vertex v and a new m-simplex $\sigma = \{v_0, \ldots, v_{m-1}, v\}$ are created which prevent s_d from being a boundary face and add σ to the coboundary of ϕ. However, if we add any of the other faces of σ to ϕ (which are all boundary faces), we obtain a new cochain ϕ' with $\delta\phi' = \tau$ and $\|\phi'\| = d + 1$. Thus, the depth of τ in X_{k+1} is at most $d + 1$.

Conversely, if τ has depth d' in X_{k+1} and $\psi = \sum_{i=1}^{d'} t_i$ is a depth-attaining cochain for τ with $\{t_i\}_{i=1}^{d'}$ satisfying the conditions in Lemma 5, then $\psi' = \sum_{i=1}^{d'-1} t_i$

is a cochain in X_k with $\delta\psi' = \tau$, so that the depth of σ is at most $d' - 1$. Thus, if τ has depth d in X_k then its depth in X_{k+1} must be at least $d+1$. Combined with the above result we conclude that all m-simplexes originally in X_k have their depth increased by exactly 1 in X_{k+1}. $\square$

Lemma 16. *$h_m(X_k) \geq m-1$ for all k.*

Proof. We know that $h_m(X_1) = m+1 \geq m-1$. Now suppose $h_m(X_k) \geq m-1$. Any chain $\phi \in C_m(\mathbb{Z}_2; X_{k+1})$ attaining h_m can be decomposed into a chain $\psi \in C_m(\mathbb{Z}_2; X_k)$ plus a chain ψ' which is a sum of depth 1 simplexes in X_{k+1}. Then we can write $\|\partial\phi\| = \|\partial\psi\| + \|\partial\psi'\| - 2x$ where x is the number of $(m-1)$-simplexes shared by $\partial\psi$ and $\partial\psi'$. Since m of the $m+1$ faces of any m-simplex in ψ' are boundary faces, $x \leq \|\psi'\|$. Also, it is clear that $\|\partial\psi'\| = (m+1)\|\psi'\|$. Thus,

$$\begin{aligned}
\frac{\|\partial\phi\|}{\|\phi\|} &= \frac{\|\partial\psi\| + (m+1)\|\psi'\| - 2x}{\|\psi\| + \|\psi'\|} \\
&\geq \frac{\|\partial\psi\| + (m-1)\|\psi'\|}{\|\psi\| + \|\psi'\|} \\
&\geq \min\left\{\frac{\|\partial\psi\|}{\|\psi\|}, m-1\right\} \\
&\geq m-1
\end{aligned}$$

(In fact, with some effort it can be seen that $h_m = \frac{(m+1)(m-1)}{(m+1)-2m^{-k+1}}$.) $\square$

By Theorem 8, $\lambda^{m-1}(X_k) = \lambda_m(X_k) \geq \frac{(m-1)^2}{2(m+1)}$. This completes the proof of part (1) of Proposition 12.

In order to define the family Y_k we need to make use of the notion of stellar subdivision, which can be traced back to at least Alexander (1930).

Definition 17 (Stellar Subdivision)**.** *Let Y be a simplicial m-complex and let $\sigma = \{v_0, \ldots, v_m\} \in S_m(Y)$. The stellar subdivision of Y along σ, denoted by $\mathrm{sd}_\sigma Y$, is* *the simplicial m-complex obtained from Y by creating a new vertex w and replacing σ with the m-simplexes*

$$\tau_i = \{v_0, \ldots, v_{i-1}, w, v_{i+1}, \ldots, v_m\}$$

where $i = 0, \ldots, m$. For notational purposes, we denote the j-th face of τ_i by $t_{i,j} := \tau_i \setminus \{v_j\}$ for $i \neq j$, and $t_{i,i} := \tau_i \setminus \{w\}$. If $\sigma_1, \ldots, \sigma_n \in S_m(Y)$, then we define the stellar subdivision of Y along the σ_i to be

$$\operatorname{sd}_{\sigma_1,\ldots,\sigma_n} Y := \operatorname{sd}_{\sigma_1} \operatorname{sd}_{\sigma_2} \cdots \operatorname{sd}_{\sigma_n} Y$$

We now define the Y_k recursively. Let Σ^m be the simplicial complex induced by a single m-simplex σ and let $Y_1 := \operatorname{sd}_\sigma \Sigma^m$. Label the m-simplexes of Y_1 as $\sigma_0, \ldots, \sigma_m$ and call their common vertex (the one created by stellar subdivision) the central vertex v. Now, given a Y_k containing the central vertex v, we call all m-simplexes containing v the inner m-simplexes of Y_k and label them as $\sigma_0, \ldots, \sigma_n$. All non-inner m-simplexes will be referred to as outer m-simplexes. We then define $Y_{k+1} := \operatorname{sd}_{\sigma_0,\ldots,\sigma_n} Y_k$. Note that v and all outer m-simplexes (and the simplexes they contain) are preserved unchanged in going from Y_k to Y_{k+1} while all of the inner m-simplexes are subdivided. Furthermore, it is clear that all the Y_k are subdivisions of Σ^m and are thus simplicial m-balls. A picture of the first few iterations of Y_k for $m = 2$ can be seen in Figure 3.10.

To prove part (2) of Proposition 12, we need to keep track of how the Cheeger numbers $h^{m-1}(Y_k)$ and $h_m(Y_k)$ change with k. This is accomplished in the following two lemmas.

Lemma 18. *$h^{m-1}(Y_k) \geq \frac{1}{k}$ for all k.*

Proof. By Lemma 6, we can prove this by keeping track of the depths of all the m-simplexes of Y_k. For Y_1, all the m-simplexes σ_i contain a boundary face (using the

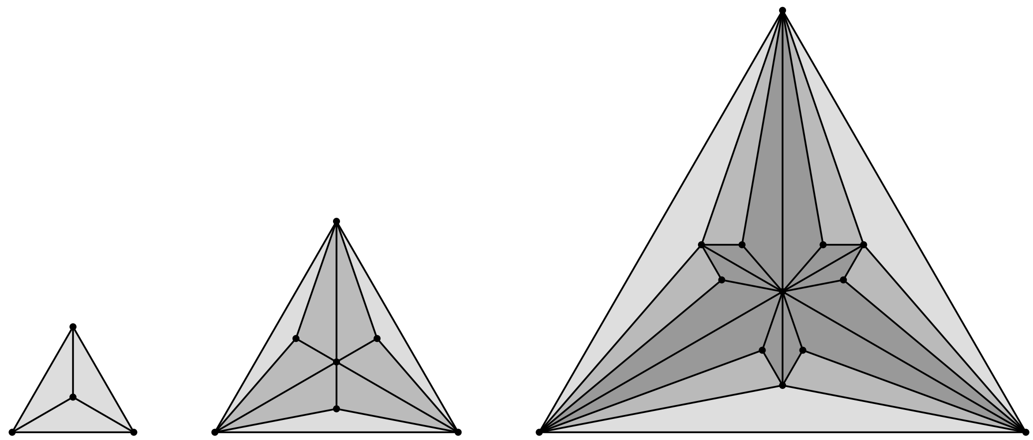

FIGURE 3.10: The first three iterations Y_1, Y_2, Y_3 of Y_k in dimension 2. The 2-simplexes have been shaded according to their depth.

notation of Definition 17 with $\sigma_i = \tau_i$, the boundary face of σ_i is $t_{i,i}$). Thus, every σ_i has depth 1 and by Lemma 6, $h^{m-1}(Y_1) = 1$. Note that the cochain ϕ which is depth-attaining for some σ_i does not include any $(m-1)$-simplex which contains v.

Now suppose for induction that every outer m-simplex σ of Y_k has depth $\leq k$ and a depth-attaining cochain $\phi \in C^{m-1}(\mathbb{Z}_2)$ such that ϕ does not contain any face of any inner m-simplex. Then in Y_{k+1}, ϕ remains unaltered, proving that σ still has depth $\leq k$ in Y_{k+1}.

Similarly, suppose that every inner m-simplex σ of Y_k has depth $\leq k$ via a depth-attaining cochain ϕ which does not contain any $(m-1)$-simplex containing v. Then in Y_{k+1}, σ is removed and replaced by new m-simplices. Using the notation of Definition 17, in Y_{k+1} the coboundary of ϕ becomes $\delta\phi = \tau_{m+1}$, so that the depth of τ_{m+1} is at most k. Furthermore, by adding any face $t_{(m+1),j}$ to ϕ ($j \neq m+1$) we obtain a cochain ϕ' with $\delta\phi' = \tau_j$, proving that the depth of τ_j is at most $k+1$. Since ϕ' still does not contain any $(m-1)$-simplex which contains v, we are back where we started. The statement now follows by induction. □

Lemma 19. *$h_m(Y_k) \leq \frac{1}{m^{k-1}}$ for all $k > 1$.*

Proof. To prove this, we merely count the number of m-simplexes in Y_k. Note that in going from Y_k to Y_{k+1} we replace $(m+1)m^{k-1}$ inner m-simplexes with $(m+1)m^k$ inner m-simplexes. Thus, Y_{k+1} has

$$(m+1)m^k - (m+1)m^{k-1} = (m+1)(m-1)m^{k-1}$$

more m-simplexes than Y_k. Since $|S_m(Y_1)| = m+1$, this means that $|S_m(Y_k)|$ is equal to

$$(m+1) + (m+1)(m-1) + (m+1)(m-1)m + \dots \\ + (m+1)(m-1)m^{k-2} = (m+1)m^{k-1}.$$

Since Y_k has $m+1$ boundary faces, the chain ϕ containing all m-simplexes of Y_k gives the upper bound on $h_m(Y_k)$:

$$h_m(Y_k) \leq \frac{\|\partial\phi\|}{\|\phi\|} = \frac{m+1}{(m+1)m^{k-1}} = \frac{1}{m^{k-1}}.$$

$\square$

By Theorem 8, $\lambda^{m-1}(Y_k) = \lambda_m(Y_k) \leq \frac{1}{m^{k-1}}$. This completes the proof of Proposition 12.

3.3 Relation to Graphs

In Chung (2007), Fan Chung defines a normalized local Dirichlet Cheeger number and normalized local Dirichlet eigenvalue and proves an inequality between them. If one translates Fan Chung's result to the unnormalized case for graphs with vertex degree upper bounded by $m+1$, it closely resembles Theorem 8.

Translating Theorem 1 of Chung (2007) into the unnormalized setting, it reads as follows. Given a graph G we can prescribe a certain set of vertices to be the boundary vertices of the graph. Let S be the prescribed boundary vertex set, and let

$$h_S := h_S(G) = \min \frac{\|\delta\phi\|}{\|\phi\|}$$

where the minimum is taken over all nonzero $\phi \in C^0(\mathbb{Z}_2)$ such that ϕ does not include any boundary vertex. Similarly, let

$$\lambda_S = \min \frac{\|\delta f\|_2^2}{\|f\|_2^2}$$

where the minimum is taken over all nonzero $f \in C^0(\mathbb{R})$ such that $f(s) = 0$ for all $s \in S$. We can also characterize λ_S as the smallest eigenvalue of L_0^S, the submatrix of L_0 consisting of the rows and columns of L_0 not indexed by vertices in S. In this case, L_0^S is a map on $C_S^0(\mathbb{R})$, the subspace of $C^0(\mathbb{R})$ spanned by the vertices not in S. Then if every vertex has degree upper bounded by $m + 1$

$$h_S \geq \lambda_S \geq \frac{h_S^2}{2(m+1)}.$$

To relate the above inequality to the simplicial complex setting, we note that for every non-branching simplicial m-complex X, one can construct a graph G (similar to the dual graph defined in Fomin et al. (2008)) as follows. Begin by constructing the simplicial complex X' as in the proof of Theorem 8 and let S be the set of border facets of X'. Create a vertex in G for every m-simplex in X'. We will use S to denote both the border facets of X' and the set of vertices in G which correspond the border facets. Connect two vertices with an edge whenever the corresponding m-simplexes are lower adjacent in X'. Since X' is non-branching, the vertices of G have degree upper bounded by $m + 1$. Identifying $C_S^0(G; \mathbb{R})$ with $C_m(X; \mathbb{R})$, we can ask if $L_m : C_m(X; \mathbb{R}) \to C_m(X; \mathbb{R})$ and $L_0^S : C_S^0(G; \mathbb{R}) \to C_S^0(G; \mathbb{R})$ are the same map. They are the same if and only if X is orientable (this is easy to see from the characterization of the Laplacian in Muhammad and Egerstedt (2006)). In addition, $h_m(X)$ and h_S are equal regardless of orientability. Thus, for non-branching orientable simplicial m-complexes, Theorem 8 reduces to the result proved by Fan Chung, and the proofs are identical. The difference is that Theorem 8 covers the more general cases of non-orientable and branching simplicial m-complexes, for which parts of the inequality may still hold.

The real projective plane provides a simple example of how orientation plays a role in our analysis of the Cheeger inequality and why it doesn't play a role in Chung (2007). In Figure 3.11, the first image shows the fundamental polygon that defines $\mathbb{R}P^2$, the second image shows a triangulation X of $\mathbb{R}P^2$, and the third image is the dual graph G of the triangulation (in the second and third image, edges with similar color are identified). In this simple example, there is no boundary ($S = \emptyset$). In the triangulation, if one considers the 2-chain $\phi \in C_2(\mathbb{Z}_2)$ which contains every 2-simplex, then $\partial\phi = 0$ and thus $h_2(X) = 0$. However, if one considers the 2-chain $f \in C_2(X; \mathbb{R})$ that assigns a 1 to every 2-simplex with the orientation shown in the figure, the boundary of f is a 1-chain which assigns a 2 to every colored edge with the orientation shown. In particular, $\partial f \neq 0$ and in fact $\lambda_2 \neq 0$ as a result of the nonorientability of $\mathbb{R}P^2$. However, the dual graph cannot see this nonorientability, as the 0-chain $\tilde{f} \in C^0_S(G; \mathbb{R})$ corresponding to f has empty coboundary, meaning $\lambda_S = 0$. Thus, in this case the map L_2 is not the same as the map L_0^S, and Theorem 1 of Chung (2007) still holds while part 1 of Theorem 8 fails.

3.4 Discussion and Open Problems

A result of the universal coefficient theorem in algebraic topology is that torsion will be an obstacle in relating higher-dimensional Cheeger numbers with eigenvalues. The Cheeger inequality for graphs holds without any assumptions since zeroth homology is never affected by torsion. For higher dimensions either the inequality does not hold or we require assumptions that remove torsion. The negative results for the

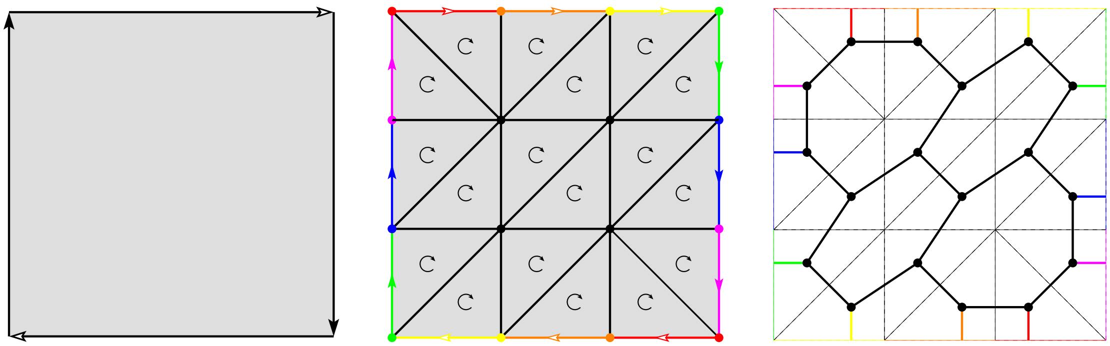

FIGURE 3.11: The fundamental polygon of $\mathbb{R}P^2$, a triangulation, and the dual graph of the triangulation.

Cheeger inequality in Gundert and Wagner (2012) are for simplicial complexes with torsion. Torsion is also known to affect algorithmic complexity. For example, the problem of finding minimal weight cycles given a simplicial complex with weights is NP-hard if there is torsion and is otherwise a linear program (see Dey et al. (2011)).

A local Cheeger number and algebraic connectivity for graphs with Dirichlet like boundary conditions was defined in Chung (2007) and a Cheeger inequality was proved. There is a close relation between Theorem 1 of Chung (2007) and Theorem 8. If Theorem 1 is adapted to an unnormalized setting (see section 3.3) then for non-branching orientable simplicial m-complexes Theorem 8 reduces to Theorem 1. However, Theorem 8 covers the more general cases of non-orientable and branching simplicial m-complexes.

We close with a few open problems of possible interest.

1. Intermediate values of k – Given a simplicial m-complex, what can we say about the relationship between h^k and λ^k or h_k and λ_k for $1 < k < m-1$? Torsion again will need to be addressed but are there some conditions under which some Cheeger-type inequalities may hold?

2. High-order eigenvalues – In Lee et al. (2011) the authors introduce higher-order (as opposed to higher-dimensional) Cheeger numbers on the graph which

correspond to higher-order eigenvalues of the graph Laplacian and prove a general Cheeger inequality for them. A natural question is how our results would extend to higher-orders. Indeed, by analogy with the Rayleigh quotient characterization of higher order eigenvalues, it would seem reasonable to define the k^{th} dimensional, j^{th} order coboundary Cheeger numbers to be

$$h^{k,j} := \min_{\substack{\phi \in C^k(\mathbb{Z}_2) \\ \phi \notin S_j}} \frac{\|\delta\phi\|}{\min_{\psi \in S_j} \|\phi + \psi\|}$$

where

$$S_j = \operatorname{span}(\operatorname{im}\delta \cup \{\phi_1, \ldots, \phi_{j-1}\})$$

is the subspace of $C^k(\mathbb{Z}_2)$ spanned by $\operatorname{im}\delta$ and cochains $\phi_1, \ldots, \phi_{j-1}$ which attain $h^{k,1}, \ldots, h^{k,j-1}$, respectively. The higher order boundary Cheeger numbers $h_{k,j}$ could be defined similarly. One would need to prove that this definition makes sense and then ask whether they satisfy any inequalities with the corresponding eigenvalues.

3. Cheeger inequalities on manifolds – Ultimately, the study of higher-dimensional Cheeger numbers on simplicial complexes should (morally speaking) be translated back to the manifold setting if possible. A tentative definition for the k-dimensional coboundary Cheeger number of a manifold M might be

$$h^k = \inf_S \frac{\mathrm{Vol}_{m-k-1}(\partial S \setminus \partial M)}{\inf_{\partial T = \partial S} \mathrm{Vol}_{m-k}(T)}$$

where Vol_k denotes k-dimensional volume and the infimum is taken over all k-codimensional submanifolds S of M. Similarly, the k-th boundary Cheeger number of M might be

$$h_k = \inf_S \frac{\mathrm{Vol}_{k-1}(\partial S)}{\inf_{\partial T = \partial S} \mathrm{Vol}_k(T)}$$

where again Vol_k denotes k-dimensional volume and the infimum is taken over all k-dimensional submanifolds S of M.

4

Stochastic Methods in the Spectral Theory of Simplicial Complexes

4.1 Introduction

4.1.1 Background

The relation between spectral graph theory and random walks on graphs has been well studied and has both theoretical and practical implications (see Chung (1997); Lovász (1996); Meilă and Shi (2001)). A classic example of this relation is graph expansion (see Hoory et al. (2006)). Loosely speaking, graph expansion measures how far a graph is from being disconnected (i.e., having a nontrivial reduced 0-th homology class). The two common characterizations of graph expansion use either the Cheeger number which relates to spectral graph theory or the mixing time of a random walk on the graph.

In this chapter we examine an analagous relation between random walks on simplicial complexes and spectral properties of higher order Laplacians. A simplicial complex is a higher-dimensional generalization of a graph consisting of vertices and edges as well as higher-dimensional simplices such as triangles and tetrahedra. The graph Laplacian was generalized to simplicial complexes by Eckmann in Eckmann (1945), resulting in what are called higher order combinatorial Laplacians. The k-th order combinatorial Laplacian, or k-Laplacian, can be used to study expansion in the sense that the spectrum of the k-Laplacian provides information on how far from the complex is from having a nontrivial k-th (co)homology class. The graph Laplacian is simply the 0-th order combinatorial Laplacian. There has been recent work extending Cheeger numbers and random walks to higher dimensions (see **?**Lubotzky (2013); Parzanchevski et al. (2012); Parzanchevski and Rosenthal (2012) and Chapter 3 of this thesis).

The k-Laplacian is naturally decomposed into two parts commonly called the up k-Laplacian and the down k-Laplacian. The graph case is an exception in that there is only an up 0-Laplacian; the down 0-Laplacian is the zero matrix. This fact suggests that a straightforward generalization of the theory of graph expansion to higher dimensions may only relate to the up k-Laplacian. Indeed, the Cheeger number of a graph was initially generalized so as to relate to the up k-Laplacian (see **?**), with the generalization to the down k-Laplacian following soon after (see Chapter 3).

This decomposition also appears when studying random walks on simplicial complexes. In a recent paper, Rosenthal and Parzanchevski generalized random walks on graphs to random walks on simplicial complexes (see Parzanchevski and Rosenthal (2012)). They defined a Markov chain on the space of oriented k-simplexes that reflects the spectrum of the up k-Laplacian, assuming $0 \leq k \leq d-1$ where d is the dimension of the simplicial complex. The walk traverses the simplicial complex by moving between oriented k-simplexes via shared $(k+1)$-simplexes. In this chapter we define a random walk that traverses the simplicial complex by traveling through shared $(k-1)$-simplexes. We demonstrate that this random walk is related to the spectrum of the down k-Laplacian and reflects the dimension of the k-th homology group over $\mathbb{R}$, assuming $1 \leq k \leq d$. We also discuss the possibility of defining other random walks on simplicial complexes, including random walks relating to the full k-Laplacian and weighted Laplacians. We also apply random walks on simplicial complexes to a semi-supervised learning problem, propagating oriented labels on edges. This generalizes the method of label propagation on graphs to a new problem in which the underlying structure to be learned is directional or is flow-like.

4.1.2 Motivation

We have two motivations for studying the random walk corresponding to the down Laplacian. The first motivation comes from an example. Consider the 2-dimensional simplicial complex formed by a hollow tetrahedron (or any triangulation of the 2-sphere). We know that the complex has nontrivial 2-dimensional homology since there is a void. However, this homology cannot be detected by the random walk defined in Parzanchevski and Rosenthal (2012), because there are no tetrahedrons that can be used by the walk to move between the triangles. In general, the walk defined in Parzanchevski and Rosenthal (2012) can detect homology from dimension 0 to co-dimension 1, but never co-dimension 0. Hence, a new walk which can travel from triangles to triangles through edges is needed.

The second motivation relates to the geometry of random walks or diffusions and manifolds. The geometry captured by the graph Laplacian as well as the Cheeger number and random walks on the graph have direct connections to the geometry of a manifold with Neumann boundary conditions. We will examine random walks that have connections to the geometry of a manifold with Dirichlet boundary conditions, denoted as "Dirichlet" random walks. Work by Fan Chung in Chung (2007) has shown that there are alternative notions of the Laplacian and random walks on graphs that capture a Dirichlet-flavored geometry of graphs. The definition of the "local" Cheeger number of a graph given in Chung (2007) bears a striking resemblance to the definition of the Cheeger number of a manifold with Dirichlet boundary (see Cheeger (1970)). Also defined in Chung (2007) is a “local” random walk that satisfies a Dirichlet boundary condition. In contrast, the usual random walk on a graph might be called Neumann. The random walk defined by Rosenthal and Parzanchevski in Parzanchevski and Rosenthal (2012) generalizes the Neumann random walk to higher dimensions on simplicial complexes. In this chapter we generalize the Dirichlet random walk.

4.1.3 Summary of Results

In this section we give a short summary of the main results. Precise definitions of the terms used are given in section 4.2.

In section 4.3 we define a p-lazy Dirichlet random walk on the oriented k-simplexes of a d-dimensional simplicial complex X, where $1 \leq k \leq d$. This walk has a corresponding probability transition matrix P. In most analyses of random walks the questions of interest are convergence and rates of convergence of $\lim_{n\to\infty} P^n\nu = \pi$, where ν is the initial probability distribution on the states, $P^n\nu$ is the marginal distribution after n steps of the walk, and π is the stationary or invariant distribution. For the usual random walk on a graph, the graph Laplacian is used to study the limiting behavoir of $P^n\nu$. For the random walks we consider, orientation issues prevent a straightforward connection between the k-Laplacian and $P^n\nu$. Instead, we find a connection between the k-Laplacian and $CTP^n\nu$ where C is a constant and T is a linear transformation. The linear transformation T enforces antisymmetry between the opposite orientations of a simplex. Denoting σ_+ and σ_- as the (arbitrarily chosen) positive and negative orientations of a simplex σ, $TP^n\nu$ is a function on the set of positively oriented simplexes such that

$$TP^n\nu(\sigma_+) = P^n\nu(\sigma_+) - P^n\nu(\sigma_-).$$

The constant C is a normalizing constant that ensures $CTP^n\nu$ has nontrivial limiting behavior. Letting M denote the maximum number of k-simplexes any $(k-1)$-simplex is contained in,

$$C = \frac{M-1}{p(M-2)+1}.$$

Let $\mathbf{1}_\tau$ denote the initial distribution supported on the oriented simplex τ and let $\widetilde{\mathcal{E}}_n^\tau := CTP^n\mathbf{1}_\tau$. The down k-Laplacian is $L_k^{\text{down}} = \delta^{k-1}\partial_k$ where δ is a coboundary operator and ∂ is the boundary operator, and let λ_k denote the smallest eigenvalue of L_k^{down} with eigenvector perpendicular to $\operatorname{im}\partial_{k+1}$. The following proposition is a direct result of Theorem 30.

Proposition 20. *If $\frac{M-2}{3M-4} < p < 1$, then the limit $\widetilde{\mathcal{E}}_\infty^\tau := \lim_{n\to\infty}\widetilde{\mathcal{E}}_n^\tau$ exists for all initial τ. In this case, the k-th homology group of X with coefficients in $\mathbb{R}$ is trivial if and only if $\widetilde{\mathcal{E}}_\infty^\tau \in \operatorname{im}\partial_{k+1}$ for all τ. In addition, if $p \geq \frac{1}{2}$ then*

$$\|\widetilde{\mathcal{E}}_n^\tau - \widetilde{\mathcal{E}}_\infty^\tau\|_2 = O\left(\left[1 - \frac{1-p}{(p(M-2)+1)(k+1)}\lambda_k\right]^n\right).$$

One difference in the above result with standard results on Markov chains is that the limiting object provides information on the homology of X. This will be discussed further in section 4.3. Another difference is that for a connected graph the random walk is irreducible, and the limit distribution is independent of the initial distribution. In higher dimensions, this independence is lost, even for complexes with trivial k-th homology over $\mathbb{R}$.

4.1.4 Related Work

The relation between graph random walks and the geometry of graphs has been examined in both Dodziuk (1984) and Chung (2007). In section 4.6.1 we show that under certain conditions the Dirichlet random walk in codimension 0 coincides with the notion of a random walk on a graph with Dirichlet boundary. A natural question to ask concerning random walks on simplicial complexes is: what would be the analogous process on manifolds? In general we are not aware of results on the continuum limit of these walks. However, the Dirichlet random walk in codimension zero is analogous to the concept of Brownian motion with killing as described by Lawler and Sokal in Lawler and Sokal (1988).

4.2 Definitions

In this section we define the simplicial complex X, the chain and cochain complexes, and the k-Laplacian.

4.2.1 Simplicial Complexes

By a simplicial complex we mean an abstract finite simplicial complex. Simplicial complexes generalize the notion of a graph to higher dimensions. Given a set of vertices V, any nonempty subset $\sigma \subseteq V$ of the form $\sigma = \{v_0, v_1, \dots, v_j\}$ is called a j-dimensional simplex, or j-simplex. A simplicial complex X is a finite collection of simplexes of various dimensions such that X is closed under inclusion, i.e., $\tau \subseteq \sigma$ and $\sigma \in X$ implies $\tau \in X$. While we will not need it for this chapter, one can include the empty set in X as well (thought of as a (-1)-simplex). Given a simplicial complex X, denote the set of j-simplexes of X as X^j. We say that X is d-dimensional or that X is a d-complex if $X^d \neq \emptyset$ but $X^{d+1} = \emptyset$. Graphs are 1-dimensional simplicial complexes. We will assume throughout that X is a d-complex for some fixed $d \geq 1$.

If $\sigma \in X^j$ and $\tau \in X^{j-1}$ and $\tau \subset \sigma$, then we call τ a *face* of σ and σ a *coface* of τ. Every j-simplex has exactly $j+1$ faces but may have any number of cofaces. Given $\sigma \in X^j$ we define $\deg(\sigma)$ (called the *degree* of σ) to be the number of cofaces of σ. Two simplexes are *upper adjacent* if they share a coface and *lower adjacent* if they share a face. The number of simplexes upper adjacent to a j-simplex σ is

$(j+1) \cdot \deg(\sigma)$ while the number of simplexes lower adjacent to σ is $\sum_{\tau \subset \sigma}(\deg(\tau) - 1)$ where the sum is over all faces τ of σ.

Orientation plays a major role in the geometry of a simplicial complex. For $j > 0$, an orientation of a j-simplex σ is an equivalence class of orderings of its vertices, where two orderings are equivalent if they differ by an even permutation. Notationally, an orientation is denoted by placing one of its orderings in square brackets, as in $[v_0, \ldots, v_j]$. Every j-simplex σ has two orientations which we think of as negatives of each other. We abbreviate these two orientations as σ_+ and $\sigma_- = -\sigma_+$ (which orientation σ_+ corresponds to is chosen arbitrarily). For $j = 0$ there are no distinct orderings, but it is useful to think of each vertex v as being positively oriented by default (so, $v_+ = v$) and having an oppositely-oriented counterpart $v_- := -v$. For any j, we will use $X^j_+ = \{\sigma_+ : \sigma \in X^j\}$ to denote a choice of positive orientation σ_+ for each j-simplex σ. The set of all oriented j-simplexes will be denoted by $X^j_\pm$, so that $X^j_\pm = \{\sigma_\pm : \sigma_+ \in X^j_+\}$ and $|X^j_\pm| = 2|X^j|$ for any choice of orientation X^j_+.

An oriented simplex $\sigma_+ = [v_0, \ldots, v_j]$ induces an orientation on the faces of σ as $(-1)^i[v_0, \ldots, v_{i-1}, v_{i+1}, \ldots, v_j]$. Conversely, an oriented face $(-1)^i[v_0, \ldots, v_{i-1}, v_{i+1}, \ldots, v_j]$ of σ induces an orientation $\sigma_+ = [v_0, \ldots, v_j]$ on σ. Two oriented j-simplexes σ_+ and σ'_+ are said to be *similarly oriented*, and we write $\sigma_+ \sim \sigma'_+$, if σ and σ' are distinct, lower adjacent j-simplexes and σ_+ and σ'_+ induce the opposite orientation on the common face (if σ and σ' are upper adjacent as well, this is the same as saying that σ_+ and σ'_+ induce the same orientation on the common coface). If they induce the same orientation on the common face, then we say they are *dissimilarly oriented* and write $\sigma_- \sim \sigma'_+$. We say that a d-complex X is *orientable* if there is a choice of orientation X^d_+ such that for every pair of lower adjacent simplexes $\sigma, \sigma' \in X^d$, the oriented simplexes $\sigma_+, \sigma'_+ \in X^d_+$ are similarly oriented.

4.2.2 Chain and Cochain Complexes

Given a simplicial complex X, we can define the chain and cochain complexes of X over $\mathbb{R}$. The space of j-chains $C_j := C_j(X;\mathbb{R})$ is the vector space of linear combinations of oriented j-simplexes with coefficients in $\mathbb{R}$, with the stipulation that the two orientations of a simplex are negatives of each other in C_j (as implied by our notation). Thus, any choice of orientation X^j_+ provides a basis for C_j. The space of j-cochains $C^j := C^j(X;\mathbb{R})$ is then defined to be the vector space dual to C_j. These spaces are isomorphic and we will make no distinction between them. Usually, we will work with cochains using the basis elements $\{\mathbf{1}_{\sigma_+} : \sigma_+ \in X^j_+\}$, where $\mathbf{1}_{\sigma_+} : C_j \to \mathbb{R}$ is defined on a basis element $\tau_+ \in X^j_+$ as

$$\mathbf{1}_{\sigma_+}(\tau_+) = \begin{cases} 1 & \tau_+ = \sigma_+ \\ 0 & \text{else} \end{cases}.$$

The boundary map $\partial_j : C_j \to C_{j-1}$ is the linear map defined on a basis element $[v_0, \ldots, v_j]$ as

$$\partial_j[v_0, \ldots, v_j] = \sum_{i=0}^{j} (-1)^i [v_0, \ldots, v_{i-1}, v_{i+1}, \ldots, v_j]$$

The coboundary map $\delta^{j-1} : C^{j-1} \to C^j$ is then defined to be the transpose of the boundary map. In particular, for $f \in C^{j-1}$,

$$(\delta^{j-1} f)([v_0, \ldots, v_j]) = \sum_{i=1}^{j} (-1)^i f([v_0, \ldots, v_{i-1}, v_{i+1}, \ldots, v_j]).$$

When there is no confusion, we will denote the boundary and coboundary maps by ∂ and δ. It holds that $\partial\partial = \delta\delta = 0$, so that (C_j, ∂_j) and (C^j, δ^j) form chain and cochain complexes.

The homology and cohomology vector spaces of X over $\mathbb{R}$ are

$$H_j := H_j(X;\mathbb{R}) = \frac{\ker \partial_j}{\operatorname{im} \partial_{j+1}} \quad \text{and} \quad H^j := H^j(X;\mathbb{R}) = \frac{\ker \delta^j}{\operatorname{im} \delta^{j-1}}.$$

It is known from the universal coefficient theorem that H^j is the vector space dual to H_j. Reduced (co)homology can also be used, and it is equivalent to including the nullset as a (-1)-dimensional simplex in X.

4.2.3 The Laplacian

The k-Laplacian of X is defined to be

$$L_k := L_k^{\mathrm{up}} + L_k^{\mathrm{down}}$$

where

$$L_k^{\mathrm{up}} = \partial_{k+1}\delta^k \quad \text{and} \quad L_k^{\mathrm{down}} = \delta^{k-1}\partial_k.$$

The Laplacian is a symmetric positive semi-definite matrix, as is each part L_k^{up} and L_k^{down}. From Hodge theory, it is known that

$$\ker L_k \cong H^k \cong H_k$$

and the space of cochains decomposes as

$$C^k = \operatorname{im} \partial_{k+1} \oplus \ker L_k \oplus \operatorname{im} \delta^{k-1}$$

where the orthogonal direct sum $\oplus$ is under the "usual" inner product

$$\langle f, g\rangle = \sum_{\sigma_+ \in X_+^k} f(\sigma_+)g(\sigma_+).$$

We are interested in the L_j^{down} half of the Laplacian. Trivially, $\operatorname{im} \partial_{j+1} \subseteq \ker L_j^{\mathrm{down}}$. The smallest nontrivial eigenvalue of L_k^{down} is therefore given by

$$\lambda_k = \min_{\substack{f \in C^k \\ f \perp \operatorname{im} \partial}} \frac{\|\partial f\|_2^2}{\|f\|_2^2},$$

where $\|f\|_2 := \sqrt{\langle f, f\rangle}$ denotes the Euclidean norm on C^k. A cochain f that achieves the minimum is an eigenvector of λ_k. It is easy to see that any such f is also an eigenvector of L_k with eigenvalue λ_k and that, therefore, λ_k relates to homology:

$$\lambda_k = 0 \Leftrightarrow \ker L_k \neq 0 \Leftrightarrow H^k \neq 0.$$

Remark 21. Given a choice of orientation X_+^k, L_k^{down} can be written as a matrix with rows and columns indexed by X_+^k, the entries of which are given by

$$(L_k^{\text{down}})_{\sigma'_+,\sigma_+} = \begin{cases} k+1 & \sigma'_+ = \sigma_+ \\ 1 & \sigma'_- \sim \sigma_+ \\ -1 & \sigma'_+ \sim \sigma_+ \\ 0 & \text{else} \end{cases}.$$

Changing the choice of orientation X_+^k amounts to a change of basis for L_k^{down}. If the row and column indexed by σ_+ are instead indexed by σ_-, all the entries in them switch sign except the diagonal entry. Alternatively, L_k^{down} can be characterized by how it acts on cochains:

$$L_k^{\text{down}} f(\tau_+) = (k+1) \cdot f(\tau_+) + \sum_{\sigma_- \sim \tau_+} f(\sigma_+) - \sum_{\sigma_+ \sim \tau_+} f(\sigma_+).$$

Note that since $L_k^{\text{down}} f$ is a cochain, $L_k^{\text{down}} f(\tau_-) = -L_k^{\text{down}} f(\tau_+)$.

The behavior of L_k^{down} is related to the following concepts:

Definition 22. *A d-complex X is called k-connected ($1 \leq k \leq d$) if for every two k-simplexes σ, σ' there exists a chain $\sigma = \sigma_0, \sigma_1, \ldots, \sigma_n = \sigma'$ of k-simplexes such that σ_i is lower adjacent to σ_{i+1} for all i. For a general d-complex X, such chains define equivalence classes of k-simplexes, and the subcomplexes induced by these are called the k-connected components of X.*

Definition 23. *A d-complex X is called disorientable if there is a choice of orientation X_+^d of its d-simplexes such that all lower adjacent d-simplexes are dissimilarly oriented. In this case, the d-cochain $f = \sum_{\sigma_+ \in X_+^d} \mathbf{1}_{\sigma_+}$ is called a disorientation.*

Remark 24. Disorientability was defined in Parzanchevski and Rosenthal (2012) and shown to be a higher-dimensional analogue of bipartiteness for graphs. Note that one can also define X to be k-disorientable if the k-skeleton of X (the k-complex given by the union $\bigcup_{i\le k} X^i$) is disorientable, but this can only happen when $k = d$. This is not hard to see: if $k < d$ then there exists a $(k+1)$-simplex $\sigma_+ = [v_0, \dots, v_k]$. Given any two dissimilarly oriented faces of σ_+, say, $[v_1, v_2, \dots, v_k]$ and $[v_0, v_2, \dots, v_k]$, we find that the simplex $\{v_0, v_1, v_3, \dots, v_k\}$ cannot be dissimilarly oriented to both of them simultaneously.

Lemma 25. *Let X be a d-complex, $1 \le k \le d$ and $M = \max_{\sigma\in X^{k-1}} \deg(\sigma)$.*

1. $\mathrm{Spec}(L_k^{down})$ *is the disjoint union of* $\mathrm{Spec}(L_k^{down}|_{X_i})$ *where X_i are the k-connected components of X.*
2. *The spectrum of L_k^{down} is contained in $[0, (k+1)M]$.*
3. *The kernal of L_k^{down} is exactly $\ker \partial_k = \mathrm{im}\, \partial_{k+1} \oplus \ker L_k$.*
4. *The upper bound $(k+1)M$ is attained if and only if $k = d$ and X has a d-connected component that is both disorientable and of constant $(d-1)$-degree.*

Proof. Statement (1) follows from the fact that L_k^{down} can be written as a block diagonal matrix with each block corresponding to a component X_i. Statement (3) is easy to verify.

For statement (2), let f be an eigenvector of L_k^{down} with eigenvalue λ, let X_+^k be a choice of orientation such that $f(\sigma_+) \ge 0$ for all $\sigma_+ \in X_+^k$ and suppose $f(\tau_+) =$

$\max_{\sigma_+ \in X_+^k} f(\sigma_+)$. Then by Remark 21,

$$\begin{aligned}
\lambda f(\tau_+) &= L_k^{\text{down}} f \\
&= (k+1) \cdot f(\tau_+) + \sum_{\sigma_- \sim \tau_+} f(\sigma_+) - \sum_{\sigma_+ \sim \tau_+} f(\sigma_+) \\
&\leq (k+1) \cdot f(\tau_+) + \sum_{\sigma_- \sim \tau_+} f(\sigma_+) + \sum_{\sigma_+ \sim \tau_+} f(\sigma_+) \\
&\leq (k+1) \cdot f(\tau_+) + \sum_{\sigma_- \sim \tau_+} f(\tau_+) + \sum_{\sigma_+ \sim \tau_+} f(\tau_+) \\
&\leq (k+1) \cdot f(\tau_+) + (k+1)(M-1) \cdot f(\sigma_+) \\
&\leq (k+1)M \cdot f(\tau_+)
\end{aligned}$$

where the third inequality results from the fact that any k-simplex is lower adjacent to at most $(k+1)(M-1)$ other k-simplexes. Therefore, $\lambda \leq (k+1)M$.

It now remains to prove statement (4). Looking back at the inequalities, it holds that $\lambda = (k+1)M$ only if $\sigma_- \sim \tau_+$ and $f(\sigma_+) = f(\tau_+)$ whenever σ and τ are lower adjacent, and the faces of σ all have degree M. But since $f(\sigma_+) = f(\tau_+)$, the same reasoning can be applied to $f(\sigma_+)$ for all σ lower adjacent to τ and eventually to all k-simplexes in the same k-connected component X_i. Ultimately, this implies that X_i has constant $(k-1)$-degree and is k-disorientable (and hence $k = d$).

To see that this bound is indeed attainable, consider a disorientable d-complex with constant $(d-1)$-degree M (this includes, for instance, the simplicial complex induced by a single d-simplex). Let X_+^d be a choice of orientation such that all lower adjacent d-simplexes are dissimilarly oriented. Then a disorientation f on X^d will satisfy

$$
\begin{aligned}
L_k^{\text{down}} f(\tau_+) &= (k+1)\cdot f(\tau_+) + \sum_{\sigma_-\sim\tau_+} f(\sigma_+) - \sum_{\sigma_+\sim\tau_+} f(\sigma_+) \\
&= (k+1)\cdot f(\tau_+) + \sum_{\sigma_-\sim\tau_+} f(\sigma_+) \\
&= (k+1)\cdot 1 + \sum_{\sigma_-\sim\tau_+} 1 \\
&= (k+1)M \cdot 1 = (k+1)M \cdot f(\tau_+)
\end{aligned}
$$

for every τ_+. □

4.3 Random walks and the k-Laplacian

In this section we define the p-lazy Dirichlet k-walk on X and relate this walk to the spectrum of the k-Laplacian.

Random walks and L_k^{down} Let X be a d-complex, $1 \leq k \leq d$, $0 \leq p < 1$, and $M = \max_{\sigma\in X^{k-1}} \deg(\sigma)$.

Definition 26. *The p-lazy Dirichlet k-walk on X is an absorbing Markov chain on the state space $S = X_\pm^k \cup \{\Theta\}$ defined as follows:*

- *Let two oriented k-cells $s, s' \in X_\pm^k$ be called textitneighbors (denoted $s \sim s'$) if they share a face and are similarly oriented. In what follows, Θ will be used to represent an additional absorbing state, called the "death state", that the Markov chain can occupy.*
- *Starting at an initial oriented k-simplex $\tau_+ \in X_\pm^k$, the walk proceeds as a time-homogenous Markov chain on the state space $S = X_\pm^k \cup \{\Theta\}$ with transition*

probabilities

$$Prob(\sigma_+ \to \sigma'_+) = Prob(\sigma_- \to \sigma'_-) = \begin{cases} p & \sigma'_+ = \sigma_+ \\ \frac{1-p}{(M-1)(k+1)} & \sigma'_+ \sim \sigma_+ \\ 0 & else, \end{cases}$$

$$Prob(\sigma_+ \to \sigma'_-) = Prob(\sigma_- \to \sigma'_+) = \begin{cases} \frac{1-p}{(M-1)(k+1)} & \sigma'_- \sim \sigma_+ \\ 0 & else, \end{cases}$$

$$Prob(\sigma_+ \to \Theta) = Prob(\sigma_- \to \Theta) = 1 - \sum_{\sigma'_+} Prob(\sigma_+ \to \sigma'_+),$$

$$Prob(\Theta \to \Theta) = 1$$

for all $\sigma, \sigma' \in X^k$*.*

- *This walk can be interpreted as follows. Starting at* τ_+*, the walk has probability* p *of staying put and for each of the neighbors of* τ_+ *the walk has probability* $\frac{1-p}{(M-1)(k+1)}$ *of jumping to that neighbor. Note that if the number of neighbors of* τ_+ *is less than* $(M-1)(k+1)$*, then the sum of these probabilities is less than 1. In this case, we interpret the difference as the probability that the walker dies (i.e., the walker jumps to a death state from which it can never return). The same holds for* τ_-*.*

The left stochastic matrix for the Markov chain is a square matrix P with rows and columns indexed by the state space $S = X^k_{\pm} \cup \{\Theta\}$ such that

$$P_{s_1,s_2} = \text{Prob}(s_2 \to s_1)$$

for all $s_1, s_2 \in S$. In stochastic processes it is more common to use the right stochastic matrix P^T as the probability matrix, for us it will be more convenient to use the left stochastic matrix. An initial distribution on the state space is a column vector ν indexed by S such that all entries are non-negative and sum to 1. The general framework in stochastic processes is to study how the marginal distribution $P^n\nu$ evolves as $n \to \infty$. Indeed, one can view the Dirichlet k-walk as a Markov chain on a graph with vertex set $V = S$ and study the limiting behavior of $P^n\nu$ within the context of graph theory. However, this is not our goal. Our goal is to connect the k-walk to the k-dimensional Laplacian, and hence to the k-dimensional topology and geometry of X.

In order to connect the k-walk to L_k, we will not study the evolution of $P^n\nu$ but rather $TP^n\nu$, the image of the marginal distribution under a linear transformation T defined as follows. Given a choice of orientation $X^k_+ = \{\sigma_+ : \sigma \in X^k\}$, T is defined to be the matrix with rows indexed by X^k_+ and columns indexed by S such that

$$(T)_{\sigma_+,\sigma_+} = 1 \qquad \text{and} \qquad (T)_{\sigma_+,\sigma_-} = -1$$

for all $\sigma \in X^k$, and such that all other entries are 0. In other words, for any function $f : S \to \mathbb{R}$, Tf is the function $Tf : X^k_+ \to \mathbb{R}$ such that

$$Tf(\sigma_+) = f(\sigma_+) - f(\sigma_-).$$

The definition of T is motivated by geometry. The geometry of simplicial complexes is characterized by the space of k-cochains C^k in which $\sigma_+ = -\sigma_-$ (and for which X^k_+ is a choice of basis). Probabilistically, σ_+ and σ_- are completely separate states for the Markov chain, but geometrically we must think of them as opposite orientations of the same underlying object σ. In addition, the state Θ has no corresponding object in C^k, so T simply removes it from the system. Of course, the vector $TP^n\nu$ does not have the property that it is always a distribution (all entries nonnegative and summing to 1), but it has the advantage that it resides in C^k and can be related to L_k as follows.

Definition 27. *The propagation matrix B of the Dirichlet k-walk is defined to be a* *square matrix indexed by X_+^k with*

$$(B)_{\sigma'_+,\sigma_+} = \begin{cases} p & \sigma'_+ = \sigma_+ \\ -\frac{1-p}{(M-1)(k+1)} & \sigma'_+ \sim \sigma_+ \\ \frac{1-p}{(M-1)(k+1)} & \sigma'_- \sim \sigma_+ \\ 0 & \textit{else} \end{cases}.$$

Proposition 28. *The propagation matrix B is given by*

$$B = \frac{p(M-2)+1}{M-1} I - \frac{1-p}{(M-1)(k+1)} \cdot L_k^{down}.$$

In addition, B satisfies $TP = BT$, so that

$$TP^n\nu = B^nT\nu.$$

Proof. The first claim is straightforwardly checked using Definition 27 and Remark 21. The second claim is equivalent to the equality $TP = BT$, which we will prove as follows. If $s \in S$ and P_s is the column of P indexed by s, then the column of TP indexed by s is TP_s. Using the definition of T, the following holds

$$\begin{aligned}(TP)_{\sigma_+,s} &= TP_s(\sigma_+) \\ &= P_s(\sigma_+) - P_s(\sigma_-) \\ &= (P)_{\sigma_+,s} - (P)_{\sigma_-,s} \\ &= \begin{cases} \pm p & s = \sigma_\pm \\ \pm\frac{1-p}{(M-1)(k+1)} & s \neq \Theta \text{ and } s \sim \sigma_\pm \\ 0 & \text{else} \end{cases}.\end{aligned}$$

Similarly, note that $(BT)_{\sigma_+,s} = B(T\mathbf{1}_s)(\sigma_+)$ where $\mathbf{1}_s$ is the vector assigning 1 to $s \in S$ and 0 to all other elements in S. If $s = \Theta$, $T\mathbf{1}_s$ is the zero vector. Otherwise, if $s = \tau_\pm$ then $T\mathbf{1}_s = \pm\mathbf{1}_{\tau_+}$ and

$$\begin{aligned}
(BT)_{\sigma_+,s} &= \pm B\,\mathbf{1}_{\tau_+}(\sigma_+) \\
&= \pm(B)_{\sigma_+,\tau_+} \\
&= \begin{cases} \pm p & \tau_+ = \sigma_+ \\ \pm\frac{1-p}{(M-1)(k+1)} & \tau_+ \sim \sigma_+ \\ \mp\frac{1-p}{(M-1)(k+1)} & \tau_- \sim \sigma_+ \\ 0 & \text{else} \end{cases} \\
&= \begin{cases} \pm p & s = \sigma_\pm \\ \pm\frac{1-p}{(M-1)(k+1)} & s \sim \sigma_\pm \\ 0 & \text{else} \end{cases} .
\end{aligned}$$

This concludes the proof. □

For what follows, we define $\mathcal{E}_n^{\tau_+} := B^n\mathbf{1}_{\tau_+}$ to be the marginal difference of the p-lazy Dirichlet k-walk on X starting at τ_+. Also, let X_+^k be a choice of orientation and denote $M = \max_{\sigma\in X^{k-1}} \deg(\sigma)$.

Corollary 29.

1. *The spectrum of B is contained in $\left[2p-1, \frac{p(M-2)+1}{M-1}\right]$, with the upper bound acheived by cochains in $\ker\partial_k$ and the lower bound acheived if and only if $k = d$ and there is a disorientable d-connected component of constant $(d-1)$-degree.*

2. *If τ has a coface, then*

$$\|\mathcal{E}_n^{\tau_+}\|_2 \geq \left(\frac{p(M-2)+1}{M-1}\right)^n \frac{1}{\sqrt{k+2}}.$$

3. *If $p \neq 0, 1$ then*

$$\|\mathcal{E}_n^{\tau_+}\|_2 \leq \max\left\{|2p-1|^n, \left(\frac{p(M-2)+1}{M-1}\right)^n\right\}.$$

Proof. Statement (1) is easy to verify with the help of Lemma 25 and Proposition 28. Statement (3) follows from the inequality $\|Af\|_2 \leq \|A\|\|f\|_2$ where A is a matrix, f is a vector, and $\|A\|$ is the spectral norm on A.

It remains now to prove statement (2). If τ has a coface σ, let $f = \partial_{k+1}\mathbf{1}_{\sigma_+}$ (with σ_+ being any orientation of σ) so that $f \in \ker \partial_k$. Let $f, f_1, \ldots, f_i$ be an orthogonal basis for C^k such that $f_1, \ldots, f_i$ are eigenvectors of B with eigenvalues $\gamma_1, \ldots, \gamma_i$, and assume $\mathbf{1}_{\tau_+} = \alpha f + \alpha_1 f_1 + \ldots + \alpha_i, f_i$. Then,

$$\begin{aligned}
\|\mathcal{E}_n^{\tau_+}\|_2 &= \|B^n \mathbf{1}_{\tau_+}\|_2 && (4.1)\\
&= \|\alpha B^n f + \alpha_1 B^n f_1 + \ldots + \alpha_i B^n f_i\|_2 && (4.2)\\
&= |\alpha| \left(\frac{p(M-2)+1}{M-1}\right)^n \|f\|_2 + |\alpha_1| \gamma_1^n \|f_1\|_2 + \ldots + |\alpha_i| \gamma_i^n \|f_i\|_2 && (4.3)\\
&\geq |\alpha| \left(\frac{p(M-2)+1}{M-1}\right)^n \|f\|_2 && (4.4)\\
&= \left(\frac{p(M-2)+1}{M-1}\right)^n \left| \left\langle \frac{f}{\|f\|_2}, \mathbf{1}_{\tau_+} \right\rangle \right| && (4.5)\\
&= \left(\frac{p(M-2)+1}{M-1}\right)^n \frac{|f(\tau_+)|}{\|f\|_2} && (4.6)\\
&= \left(\frac{p(M-2)+1}{M-1}\right)^n \frac{1}{\sqrt{k+2}} && (4.7)
\end{aligned}$$

□

Note that if $p \neq 0, 1$, then $|2p-1|$ and $\frac{p(M-2)+1}{M-1}$ are both less than one. Hence, the above corollary says that the limit of the marginal difference is trivial in general. We can remove this trivial behavior by making one final alteration to our object of study: multiply the propagation matrix B by $\frac{M-1}{p(M-2)+1}$ to obtain the normalized propagation matrix $\widetilde{B} := \frac{M-1}{p(M-2)+1} B$ and define $\widetilde{\mathcal{E}}_n^{\tau_+} := \widetilde{B}^n \mathbf{1}_{\tau_+}$ to be the normalized marginal difference. The next two theorems show that the homology of X can be determined from the limiting behavior of the normalized marginal difference.

Theorem 30.

The limit $\widetilde{\mathcal{E}}_{\infty}^{\tau_+} := \lim_{n\to\infty} \widetilde{\mathcal{E}}_{n}^{\tau_+}$ *of the normalized marginal difference exists for all* τ_+ *if and only if* $\widetilde{B}$ *has no eigenvalue* $\lambda \leq -1$. *Furthermore,* $\widetilde{\mathcal{E}}_{\infty}^{\tau_+} = \mathrm{proj}_{\ker \partial_k} \boldsymbol{1}_{\tau_+}$ *whenever* $\widetilde{\mathcal{E}}_{\infty}^{\tau_+}$ *exists, where* $\mathrm{proj}_{\ker \partial_k}$ *is the projection map onto* $\ker \partial_k$.

Proof. Note that by Corollary 29, the spectrum of $\widetilde{B}$ is upper bounded by 1 and the eigenspace of the eigenvalue 1 is exactly $\ker \partial_k$. Let $f_1, \ldots, f_i$ be an orthogonal basis for C^k such that $f_1, \ldots, f_i$ are eigenvectors of $\widetilde{B}$ with eigenvalues $\gamma_1, \ldots, \gamma_i$. Then any $\mathbf{1}_{\tau_+}$ can be written as a linear combination $\mathbf{1}_{\tau_+} = \alpha_1 f_1 + \ldots + \alpha_i, f_i$ so that

$$\widetilde{\mathcal{E}}_{\infty}^{\tau_+} = \widetilde{B}^n \mathbf{1}_{\tau_+} = \alpha_1 \gamma_1^n f_1 + \ldots, \alpha_i \gamma_i^n f_i$$

Since the f_j form a basis, $\widetilde{\mathcal{E}}_{\infty}^{\tau_+}$ converges if and only if $\alpha_j \gamma_j^n$ converges for each j. In other words, $\widetilde{\mathcal{E}}_{\infty}^{\tau_+}$ converges if and only if for every j, $\alpha_j = 0$ or $\gamma_j > -1$. Furthermore, the limit (when it exists) is always

$$\sum_{\{j:\gamma_j=1\}} \alpha_j f_j = \mathrm{proj}_{\ker \partial_k} \mathbf{1}_{\tau_+}$$

Finally, suppose $\widetilde{B}$ has an eigenvalue $\lambda \leq -1$. Then there is an eigenvector f such that $\widetilde{B}^n f = \lambda^n f$ does not converge. Since the set of cochains $\{\mathbf{1}_{\tau_+} : \tau_+ \in X_{\pm}^k\}$ spans $C^k(\mathbb{R})$, f can be written as a linear combination of them and therefore $\widetilde{B}^n \mathbf{1}_{\tau_+}$ must not converge for some τ_+. □

Theorem 31.

1. *If* $\frac{M-2}{3M-4} < p < 1$ *then the limit* $\widetilde{\mathcal{E}}_{\infty}^{\tau_+}$ *exists for all* τ_+ *and*

$$\dim(\mathrm{span}\{\mathrm{proj}_{\ker \delta^k} \widetilde{\mathcal{E}}_{\infty}^{\tau_+} : \tau_+ \in X_{\pm}^k\}) = \dim(H_k(X))$$

where $\mathrm{proj}_{\ker \delta^k}$ *denotes the projection map onto* $\ker \delta^k$.

2. *The same holds when* $p = \frac{M-2}{3M-4}$ *and either* $k < d$ *or there are no disorientatable* d*-connected components of constant* $(d-1)$*-degree.*

3. *We can say more if* $p \geq \frac{1}{2}$*. In this case,*

$$\|\widetilde{\mathcal{E}}_n^{\tau_+} - \widetilde{\mathcal{E}}_\infty^{\tau_+}\|_2 = O\left(\left[1 - \frac{1-p}{(p(M-2)+1)(k+1)}\lambda_k\right]^n\right)$$

Proof. The proof follows mostly from Theorem 30. According to that theorem, $\widetilde{\mathcal{E}}_\infty^{\tau_+}$ exists for all τ_+ if and only if the spectrum of $\widetilde{B}$ is contained in $(-1, 1]$. Using Corollary 29 and the definition $\widetilde{B} := \frac{M-1}{p(M-2)+1}B$, we know that the spectrum of $\widetilde{B}$ is contained in $\left[(2p-1)\frac{M-1}{p(M-2)+1}, 1\right]$. Now,

$$(2p-1)\tfrac{M-1}{p(M-2)+1} > -1$$

$$\Updownarrow$$

$$\tfrac{p(M-2)+1}{M-1} > 1 - 2p$$

$$\Updownarrow$$

$$p\left(\tfrac{M-2}{M-1} + 2\right) > 1 - \tfrac{1}{M}$$

$$\Updownarrow$$

$$p > \tfrac{M-2}{3M-4},$$

which proves that the spectrum of $\widetilde{B}$ is indeed contained in $(-1, 1]$ when $p > \frac{M-2}{3M-4}$. Since the $\mathbf{1}_{\tau_+}$ span all of C^k, the $\widetilde{\mathcal{E}}_\infty^{\tau_+} = \operatorname{proj}_{\ker \partial_k} \mathbf{1}_{\tau_+}$ span all of $\ker \partial_k$, and hence the $\operatorname{proj}_{\ker \delta^k} \widetilde{\mathcal{E}}_\infty^{\tau_+}$ span all of $\ker L_k$.

In the case that $p = \frac{M-2}{3M-4}$, the spectrum of $\widetilde{B}$ is contained in $[-1, 1]$. However, as long as -1 is not actually an eigenvalue of $\widetilde{B}$, the result still holds. According to Corollary 29, -1 is an eigenvalue if and only if $k = d$ and there is a disorientable

d-connected component of constant $(d-1)$-degree. The case $p = 1$ is trivial ($\widetilde{B} = I$) and not considered.

Finally, if the spectrum of B lies in $(-1, 1]$ and λ is the eigenvalue of $\widetilde{B}$ contained in $(-1, 1)$ with largest absolute value, so

$$\|\widetilde{B}^n f - \lim_{n\to\infty} \widetilde{B}^n f\|_2 \leq |\lambda|^n \|f\|_2$$

for all f. Let $f_1, \ldots, f_i$ be an orthonormal basis for C^k such that $f_1, \ldots, f_i$ are eigenvectors of $\widetilde{B}$ with eigenvalues $\gamma_1, \ldots, \gamma_i$. Then any f can be written as a linear combination $f = \alpha_1 f_1 + \ldots, +\alpha_i f_i$ and so that $\|f\|_2 = \sum_j |\alpha_j|$ and

$$\begin{aligned}
\|\widetilde{B}^n f - \lim_{n\to\infty} \widetilde{B}^n f\|_2 &= \|\alpha_1 \gamma_1^n f_1 + \ldots + \alpha_i \gamma_i^n f_i - \sum_{\{j:\gamma_j = 1\}} \alpha_j f_j\|_2 \\
&= \|\sum_{\{j:\gamma_j \neq 1\}} \alpha_j \gamma_j^n f_j\|_2 \\
&= \sum_{\{j:\gamma_j \neq 1\}} |\alpha_j \gamma_j^n| \|f_j\|_2 \\
&\leq \sum_{\{j:\gamma_j \neq 1\}} |\alpha_j| |\lambda|^n \\
&\leq |\lambda|^n \|f\|_2
\end{aligned}$$

In particular, if $p \geq \frac{1}{2}$ then the spectrum of $\widetilde{B}$ is contained in $[0, 1]$ and therefore $\lambda = 1 - \frac{1-p}{(p(M-2)+1)(k+1)} \lambda_k$. □

Note the dependence of the theorem on both the lazy probability p and on M. We can think of M as the maximum amount of "branching", where $M = 2$ means there is no branching, as in a pseudomanifold of dimension $d = k$, and large values of M imply a high amount of branching. In particular, the walk must become more and more lazy for larger values of M in order to prevent the marginal difference from diverging. However, since $\frac{M-2}{3M-4} < \frac{1}{3}$ for all M a lazy probability of at least $\frac{1}{3}$

will always ensure convergence. While there is no explicit dependence on k or the dimension d, it is easy to see that M must always be at least $d-k+1$ (for instance, it is not possible for a triangle complex to have maximum vertex degree 1).

We would also like to know whether for the normalized marginal difference converges to 0. Note that if τ_+ has a coface, then we already know that $\|\mathcal{E}_n^{\tau_+}\|_2$ stays bounded away from 0 according to Corollary 29. However, if τ has no coface, then $\mathbf{1}_{\tau_+}$ may be perpendicular to $\ker \partial_k$, allowing $\|\mathcal{E}_n^{\tau_+}\|_2$ to die in the limit as we see in the following corollary.

Corollary 32. *If τ has no coface, $H_k = 0$, and if $\frac{M-2}{3M-4} < p < 1$ then*

$$\|\mathcal{E}_\infty^{\tau_+}\|_2 = 0.$$

The same is true when $p = \frac{M-2}{3M-4}$ and either $k < d$ or there are no disorientable d-connected components of constant $(d-1)$-degree,

Proof. Under all conditions stated, $\widetilde{\mathcal{E}}_\infty^{\tau_+}$ converges. If τ has no coface, then $\mathbf{1}_{\tau_+}$ is in the orthogonal complement of $\operatorname{im} \partial_{k+1}$, because all elements of $\operatorname{im} \partial_{k+1}$ are supported on oriented faces of $(k+1)$-simplexes. If $H_k = 0$ then $\ker \partial_k = \operatorname{im} \partial_{k+1}$, so that

$$\|\widetilde{\mathcal{E}}_\infty^{\tau_+}\|_2 = \operatorname{proj}_{\ker \partial_k} \mathbf{1}_{\tau_+} = 0.$$

□

4.4 Random walks with Neumann boundary conditions

The Neumann random walk described by Rosenthal and Parzanchevski in Parzanchevski and Rosenthal (2012) is the "dual" of the Dirichlet random walk, jumping from simplex to simplex through cofaces rather than faces. Let X be a d-complex, $0 \leq k \leq d-1$, and $0 \leq p < 1$.

Definition 33. *The p-lazy Neumann k-walk on X is an absorbing markov chain on the state space $S = X^k_\pm \cup \{\Theta\}$ defined as follows:*

- *Let two oriented k-simplexes $s, s' \in X^k_\pm$ be called coneighbors (denoted $s \frown s'$) if they share a coface and are dissimilarly oriented. Also, let $\deg(\sigma)$ denote the number of cofaces of σ. In what follows, Θ is an additional absorbing state the random walk can occupy, called the "death state".*

- *Starting at an initial oriented k-simplex $\tau_+ \in X^k_\pm$ the walk proceeds as a time-homogeneous Markov chain on $S := X^k_\pm \cup \{\Theta\}$ with transition probabilities*

$$Prob(\sigma_+ \to \sigma'_+) = Prob(\sigma_- \to \sigma'_-) = \begin{cases} p & \sigma'_+ = \sigma_+ \\ \frac{1-p}{k \cdot \deg(\sigma)} & \sigma'_+ \frown \sigma_+ \\ 0 & else, \end{cases}$$

$$Prob(\sigma_+ \to \sigma'_-) = Prob(\sigma_- \to \sigma'_+) = \begin{cases} \frac{1-p}{k \cdot \deg(\sigma)} & \sigma'_- \frown \sigma_+ \\ 0 & else, \end{cases}$$

$$Prob(\sigma_+ \to \Theta) = Prob(\sigma_- \to \Theta) = \begin{cases} 1-p & \deg(\sigma) = 0 \\ 0 & else \end{cases},$$

$$Prob(\Theta \to \Theta) = 1.$$

  for all $\sigma, \sigma' \in X^k$.

- *This walk can be described as follows. Starting at at any σ_+, the walk has a probability p of staying put and otherwise is equally likely to jump to one of the $k \cdot \deg(\sigma)$ coneighbors of σ_+. If σ has no coneighbors (i.e., if σ has no cofaces), then the walk instead has probability p of staying put and probability $1-p$ of jumping to the absorbing state Θ. The same holds for starting at σ_-.*

This definition varies from that in Parzanchevski and Rosenthal (2012) where the case of $k = d-1$ was examined and it was assumed that every k-simplex had at least one coface, and as a result a death state was not required. The inclusion of the death state in all cases in the definition above allows us to use the matrix T from Section 4.3 to relate the marginal distribution of the walk to L_k^{up}. If ν is an initial distribution and P is the left stochastic matrix for the walk (so that $P^n\nu$ is the marginal distribution after n steps), then $TP^n\nu$ is the marginal difference after n steps for the Neumann k-walk. Similar to the Dirichlet walk, there is a propagation matrix A such that $TP^n\nu = A^nT\nu$ and such that A relates to L_k^{up}. Once again the marginal difference converges to 0 for all initial distributions, but this behavior is fixed by multiplying A by a constant, obtaining a normalized propagation matrix $\widetilde{A}$ and a normalized marginal distribution $\widetilde{A}^nT\nu$. The limiting behavior of the normalized marginal difference reveals homology similar to Theorem 31.

While the results for the Neumann and Dirichlet walks are quite similar, we highlight two differences. One is that the norm of the normalized marginal difference for the Neumann k-walk starting at a single oriented simplex stays bounded away from 0 (see Proposition 2.8 of Parzanchevski and Rosenthal (2012)), whereas this need not hold for the Dirichlet k-walk (as in Corollary 32). This is because in the Neumann case, every starting point $\mathbf{1}_{\tau_+}$ has some nonzero inner product with an element of $\operatorname{im}\delta^{k-1} \subseteq \ker\delta^k$. The second difference is in the threshold values for p in Theorem 31 and in the corresponding Theorem 2.9 of Parzanchevski and Rosenthal (2012). For the Dirichlet walk, homology can be detected for $p > \frac{M-2}{3M-4}$ (where $M = \max_{\sigma\in X^{k-1}} \deg(\sigma)$) whereas for the Neumann walk the threshold is $p > \frac{k}{3k+2}$. Hence, the Neumann walk is sensitive to the dimension while the Dirichlet walk is sensitive to the maximum degree. In both cases, $p \geq \frac{1}{3}$ is always sufficient to detect homology and $p \geq \frac{1}{2}$ allows us to put a bound on the rate of convergence.

4.5 Other Random Walks

The examples of the Dirichlet random walk and the Neumann random walk suggest that a more general method for relating matrices to random walks is possible. So far only the unweighted Laplacian matrices L_k^{up} and L_k^{down} have been found to relate to random walks, but one might ask whether the full Laplacian matrix $L_k = L_k^{\text{up}} + L_k^{\text{down}}$ as well as weighted Laplacians can be related to random walks. Weighted Laplacians will not be considered in this dissertation, but can be defined as

$$\mathcal{L}_k = \mathcal{L}_k^{\text{up}} + \mathcal{L}_k^{\text{down}}$$

where

$$\mathcal{L}_k^{\text{up}} := W_k^{-1/2} \partial_{k+1} W_{k+1} \delta^k W_k^{-1/2} \text{ and } \mathcal{L}_k^{\text{down}} := W_k^{1/2} \delta^{k-1} W_{k-1}^{-1} \partial_k W_k^{1/2}$$

and where W_j denotes a diagonal matrix with diagonal entries equal to positive weights, one for each j-simplex. In order to make a broad theorem relating Laplacians to random walks, we introduce the following notion of an "X_+^k-matrix".

Definition 34. *Let X_+^k be a choice of orientation. An X_+^k-matrix is a square matrix L such that*

1. *the rows and columns of L are indexed by X_+^k,*
2. *L has nonnegative diagonal entries,*
3. *whenever L has a zero on the diagonal, all other entries in the same row or column are also zero.*

Definition 35. *Let X_+^k be a choice of orientation, L an X_+^k-matrix, and $p \in [0, 1]$. We define the p-lazy propagation matrix related to L to be*

$$A_{L,p} := \frac{p(K-1)+1}{K} I - \frac{1-p}{K} \cdot L D_L^{-1}$$

where $p \in [0,1]$*,* $K := \max_{\sigma_+ \in X_+^k} \sum_{\sigma'_+ \neq \sigma_+} |(LD_L^{-1})_{\sigma'_+, \sigma_+}|$*, and* D_L *is the diagonal matrix with the same nonzero diagonal entries as* L *and with all other diagonal entries equal to 1 (or any nonzero number, as property (3) of Definition 34 ensures* LD_L^{-1} *will be unchanged). The case* $K = 0$ *is degenerate and not considered. If* $(D_L)_{\sigma_+, \sigma_+} = 0$*, then* $(D_L^{-1})_{\sigma_+, \sigma_+} = 0$ *by convention. In addition, we define the normalized* p*-lazy propagation matrix relating to* L *to be*

$$\widetilde{A}_{L,p} := I - \frac{1-p}{p(K-1)+1} LD_L^{-1} \left(= \frac{K}{p(K-1)+1} A_{L,p} \right)$$

Note that whenever $K = 1$, $A_{L,p} = \widetilde{A}_{L,p}$. In particular, this is true in the graph case when $L = L_0$.

Definition 36. *Let* X_+^k *be a choice of orientation,* L *an* X_+^k*-matrix,* $p \in [0,1]$*, and let* $A_{L,p}$ *be defined as above. We define* $P_{L,p}$ *to be the square matrix with rows and columns indexed by* $S := X_+^k \cup \{\Theta\}$ *with*

$$(P_{L,p})_{\sigma'_+, \sigma_+} = (P_{L,p})_{\sigma'_-, \sigma_-} = \begin{cases} (A_{L,p})_{\sigma'_+, \sigma_+} & \textit{if } (A_{L,p})_{\sigma'_+, \sigma_+} > 0 \\ 0 & \textit{else} \end{cases},$$

$$(P_{L,p})_{\sigma'_-, \sigma_+} = (P_{L,p})_{\sigma'_+, \sigma_-} = \begin{cases} -(A_{L,p})_{\sigma'_+, \sigma_+} & \textit{if } (A_{L,p})_{\sigma'_+, \sigma_+} < 0 \\ 0 & \textit{else} \end{cases},$$

$$(P_{L,p})_{s,\Theta} = 0 \textit{ for all } s \neq \Theta,$$

$$(P_{L,p})_{\Theta,s} = 1 - \sum_{s' \in S \setminus \{\Theta\}} (P_{L,p})_{s',s} \textit{ for all } s \neq \Theta,$$

and

$$(P_{L,p})_{\Theta,\Theta} = 1.$$

The following lemma says that $P_{L,p}$ is always a probability matrix.

Lemma 37. *Let X_+^k be a choice of orientation, L an X_+^k-matrix, and $p \in [0,1]$. The matrix $P_{L,p}$ defined above is the left stochastic matrix for an absorbing Markov chain on the state space S (i.e., $(P_L)_{s',s} = Prob(s \to s')$) such that Θ is an absorbing state and $Prob(s \to s) = p$ for all $s \neq \Theta$.*

Proof. It is clear by the definition of $P_{L,p}$ that Θ is an absorbing state. To see that $\text{Prob}(s \to s) = p$ for all $s \neq \Theta$, note that

$$\begin{aligned}(A_{L,p})_{\sigma_+,\sigma_+} &= \frac{p(K-1)+1}{K} - \frac{1-p}{K} \cdot 1 \\ &= \frac{p(K-1)+1-1+p}{K} = p\end{aligned}$$

and hence by the definition of $P_{L,p}$,

$$(P_{L,p})_{\sigma_-,\sigma_-} = (P_{L,p})_{\sigma_+,\sigma_+} = p$$

for all σ. It is also clear by the definition of $P_{L,p}$ that the entries $(P_{L,p})_{\sigma'_-,\sigma_+} = (P_{L,p})_{\sigma'_+,\sigma_-}$ are nonnegative for any σ, σ'. Hence, in order to show that $P_{L,p}$ is left stochastic we need only to prove that $\sum_{s' \in S \setminus \{\Theta\}} (P_{L,p})_{s',s} \leq 1$ for all $s \in S \setminus \{\Theta\}$. By the symmetries inherent in $P_{L,p}$, the value of the sum is the same for $s = \sigma_+$ as it is for $s = \sigma_-$. For any $s = \sigma_+$,

$$\begin{aligned}\sum_{s' \in S \setminus \{\Theta\}} (P_{L,p})_{s',s} &= \sum_{\sigma'_+ \in X_+^k} (A_{L,p})_{\sigma'_+,\sigma_+} \\ &= p + \sum_{\sigma'_+ \in X_+^k \setminus \{\sigma_+\}} |(A_{L,p})_{\sigma'_+,\sigma_+}| \\ &= p + \frac{1-p}{K} \sum_{\sigma'_+ \in X_+^k \setminus \{\sigma_+\}} |(LD_L^{-1})_{\sigma'_+,\sigma_+}| \\ &\leq p + (1-p) = 1.\end{aligned}$$

This completes the proof. □

We will call $P_{L,p}$ the p-lazy probability matrix related to L. The following theorem shows that $P_{L,p}$ is related L.

Theorem 38. *Let X_+^k be a choice of orientation, L an X_+^k-matrix, $p \in [0,1]$, and let $A_{L,p}$ and $P_{L,p}$ be defined as above. In addition, let T_+ be defined as in section 4.3. Then*

$$A_{L,p}T = TP_{L,p}.$$

In other words, the evolution of the marginal differences $T_+P_{L,p}^n\nu$ after n steps with initial distribution ν is governed by the propagation matrix: $TP_{L,p}^n\nu = A_{L,p}^n T\nu$.

Proof. Using the definition of T

$$\begin{aligned}(TP_{L,p})_{\sigma_+,s} &= (P_{L,p})_{\sigma_+,s} - (P_{L,p})_{\sigma_-,s} \\ &= \begin{cases} \pm(A_{L,p})_{\sigma_+,\sigma'_+} & s = \sigma'_\pm \\ 0 & s = \Theta \end{cases}.\end{aligned}$$

Similarly, note that $(A_{L,p}T)_{\sigma_+,s} = A_{L,p}(T\mathbf{1}_s)(\sigma_+)$ where $\mathbf{1}_s$ is the vector assigning 1 to $s \in S$ and 0 to all other elements in S. If $s = \Theta$, $T\mathbf{1}_s$ is the zero vector. Otherwise, if $s = \tau_\pm$ then $T\mathbf{1}_s = \pm\mathbf{1}_{\tau_+}$. Thus,

$$\begin{aligned}(A_{L,p}T)_{\sigma_+,s} &= \begin{cases} \pm A_{L,p}\mathbf{1}_{\tau_+}(\sigma_+) & s = \tau_\pm \\ 0 & s = \Theta \end{cases} \\ &= \begin{cases} \pm(A_{L,p})_{\sigma_+,\tau_+} & s = \tau_\pm \\ 0 & s = \Theta \end{cases}.\end{aligned}$$

This concludes the proof. □

Finally, we conclude with a few results motivating the normalized propagation matrix and showing how the limiting behavior of the marginal difference relates to the kernel and spectrum of L. We strongly suspect stronger results hold.

Theorem 39. *Let X_+^k be a choice of orientation, L an X_+^k-matrix with Spec$(L) \subset [0,\Lambda]$ $(\Lambda > 0)$. Then for $\frac{\Lambda-1}{K+\Lambda-1} \le p < 1$ the following statements hold:*

1. $\|A_{L,p}^n T\nu\|_2 \to 0$ *for every initial distribution* ν,

2. $\widetilde{A}_{L,p}^n T\nu \to \text{proj}_{\ker LD^{-1}} T\nu$ *for every initial distribution* ν, *where* $\text{proj}_{\ker L}$ *denotes the projection map onto the kernel of* L,

3. *If* λ *is the spectral gap (smallest nonzero eigenvalue) of* L *then*

$$\|\widetilde{A}_{L,p}^n T\nu - \text{proj}_{\ker L} T\nu\|_2 = O\left(\left[1 - \frac{1-p}{p(K-1)+1}\lambda\right]^n\right).$$

Proof. The proof is the same as in the proofs of Corollary 29 and Theorem 31 and mostly boil down to statements about the spectra of $A_{L,p}$ and $\widetilde{A}_{L,p}$. Note that since $\frac{\Lambda-1}{K+\Lambda-1} \leq p < 1$, $\text{Spec}(\widetilde{A}_{L,p}) \subset [0,1]$ where the eigenspace of the eigenvalue 1 is equal to the kernel of L, and the largest eigenvalue of $\widetilde{A}_{L,p}$ less than 1 is $1 - \frac{1-p}{p(K-1)+1}\lambda$. $\square$

As an example of the applicability of this framework, $\widetilde{A}_{L,p}$ is used with $L = L_k$ to perform label propagation on edges in the next section.

4.6 Examples of random walks

In this section we state some specific random walks to provide some intuition for random walks on complexes and to use the ideas we have developed to study a problem in machine learning, semi-supervised learning.

4.6.1 Triangle complexes

We begin by reviewing local random walks on graphs as defined by Fan Chung in Chung (2007). Given a graph $G = (V, E)$ and a designated "boundary" subset $S \subset V$, a $\frac{1}{2}$-lazy random walk on $\overline{S} = V \setminus S$ can be defined which satisfies a Dirichlet boundary condition on S (meaning a walker is killed whenever it reaches S). The walker starts on a vertex $v_0 \in \overline{S}$ and at each step remains in place with probability $\frac{1}{2}$ or else jumps to one of the adjacent vertices with equal probability. The boundary condition is enforced by declaring that whenever the walker would jump to a vertex in S, the walk ends. Thus, the left stochastic matrix P for this walk can be written down as

$$(P)_{v',v\in\overline{S}} = \text{Prob}(v \to v') = \begin{cases} \frac{1}{2} & \text{if } v = v' \\ \frac{1}{2d_v} & \text{if } v \sim v' \\ 0 & \text{else} \end{cases}$$

where $v \sim v'$ denotes that vertices v and v' are adjacent and d_v is the number of edges connected to v. Note that P is indexed only by $\overline{S}$, and that its columns sums may be less than 1. The probability of dying is implicitly encoded in P as the difference between the column sum and 1. As was shown in Chung (2007), P is related to a local Laplace operator also indexed by $\overline{S}$. If D is the degree matrix and A the adjacency matrix, the graph Laplacian of G is $L = D - A$. We denote the local Laplacian as L_S, where S in subscript means rows and columns indexed by S have been deleted. The relation between P and L_S is

$$P = I - \frac{1}{2}L_S D_S^{-1}.$$

Hence, the existence and rate of convergence to a stationary distributions can be studied in terms of the spectrum of the local Laplace operator.

Now suppose we are given an orientable 2-dimensional non-branching simplicial complex $X = (V, E, T)$ where T is the set of triangles (subsets of V of size 3). Non-branching means that every edge is contained in at most 2 triangles. We can define a random walk on triangles fundamentally identical to a local walk on a graph which reveals the 2-dimensional homology of X. The $\frac{1}{2}$-lazy Dirichlet 2-walk on T starts at a triangle t_0 and at each step remains in place with probability $\frac{1}{2}$ or else jumps to the other side of one of the three edges. If no triangle lies on the other side of the

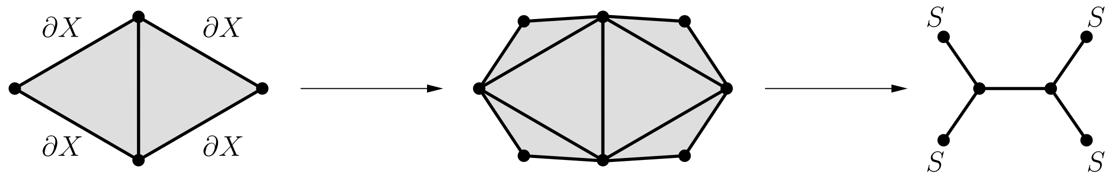

FIGURE 4.1: Making the Dirichlet boundary condition explicit, and translating into a graph.

edge, the walk ends. The transition matrix B for this walk is given by

$$(B)_{t',t} = \text{Prob}(t \to t') = \begin{cases} \frac{1}{2} & \text{if } t = t' \\ \frac{1}{6} & \text{if } t \sim t' \\ 0 & \text{else} \end{cases}$$

where $t \sim t'$ denotes t and t' share an edge. This is the same transition matrix as P, in the case that $d_v = 3$ for all $v \in \overline{S}$. In this case, the analog of the set S is the set of edges that are contained in only one triangle, which is the boundary of X. To draw an explicit connection, imagine adding a triangle to each boundary edge, obtaining a larger complex $\widetilde{X} = (\widetilde{V}, \widetilde{E}, \widetilde{T})$. See Figure 4.1

Then take the "dual graph" $G = (V, E)$ of $\widetilde{X}$ by thinking of triangles as vertices (so, $V = \widetilde{T}$) and connecting vertices in G with an edge if the corresponding triangles in $\widetilde{X}$ share an edge. Choose the vertices corresponding to the added triangles $\widetilde{T} \setminus T$ to be the boundary set S. Now the matrix P associated to the local random walk on G is indistinguishable from the matrix B associated to the random walk on X. In addition, it can be seen that L_S on G is the same as L_2, the 2-dimensional Laplacian on X defined with respect to a given orientation we have assumed orientability assumption). The following states the relation between the transition matrices and Laplacians:

$$B = P = I - \frac{1}{6}L_S = I - \frac{1}{6}L_2.$$

See section 4.2 for the definition of L_2, and Chapter 3 of this thesis for more on the connection between L_S and L_2.

It is a basic fact that the kernel of L_2 corresponds to the 2-dimensional homology group of X over $\mathbb{R}$. Therefore, there exists a stationary distribution for the random walk if and only if X has nontrivial homology in dimension 2. Additionally, the rate of convergence to the stationary distribution (if it exists) is governed by the spectral gap of L_2. In particular, the following statements hold:

1. Given a starting triangle t_0, the marginal distribution of the random walk after n steps is $\mathcal{E}_n^{t_0} := B^n \mathbf{1}_{t_0}$ where $\mathbf{1}_{t_0}$ is the vector assigning a 1 to t_0 and 0 to all other triangles. For any t_0, the marginal distrubition converges, i.e., $\mathcal{E}_\infty^{t_0} := \lim_{n\to\infty} \mathcal{E}_n^{t_0}$ exists.

2. The limit $\mathcal{E}_\infty^{t_0}$ is equal to 0 for all starting triangles t_0 if and only if X has trivial homology in dimension 2 over $\mathbb{R}$.

3. The rate of convergence is given by

$$\|\mathcal{E}_n^{t_0} - \mathcal{E}_\infty^{t_0}\|_2 = O\left(\left[1 - \frac{1}{6}\lambda_2\right]^n\right)$$

   where λ_2 is the smallest nonzero eigenvalue of L_2.

The example given here is constrained by certain assumptions (orientability and the non-branching property), which allows for the most direct interpretation with respect to previous work done on graphs.

4.6.2 Label propagation on edges

In machine learning random walks on graphs have been used for semi-supervised learning. In this section we will generalize a class of algorithms on graphs called "label propogation" algorithms to simplicial complexes, specifically we extend the algorithm described in Zhu et al. (2005) (for more examples, see Callut et al. (2008); Jaakkola and Szummer (2002); Zhou and Schölkopf (2004)). The goal of semi-supervised classification learning is to classify a set of unlabelled objects $\{v_1, \ldots, v_u\}$, given a small set of labelled objects $\{v_{u+1}, \ldots, v_{u+\ell}\}$ and a set E of pairs of objects $\{v_i, v_j\}$ that one believes *a priori* to share the same class. Let $G = (V, E)$ be the graph with vertex set $V = \{v_1, \ldots, v_{u+\ell}\}$ and let P be the probability matrix for the usual random walk, i.e.,

$$(P)_{ij} = \text{Prob}(v_j \to v_i) = \frac{1}{d_j}$$

where d_j is the degree of vertex j. We denote the classes an object belongs to as $c = 1, ..., C$ and an initial distribution $f_0^c : V \to [0, 1]$ is the *a priori* confidence that each vertex is in class c, a recursive label propagation process proceeds as follows.

1. For $t = 1, ..., T$ and $c = 1, .., C$:

   (a) Set $f_t^c \leftarrow P f_{t-1}^c$

   (b) Reset $f_t^c(v_i) = 1$ for all v_i labelled as c.

2. Consider f_T^c as an estimate of the relative confidence that each object is in class c.

3. For each unlabelled point v_i, $i \leq u$, assign the label

$$\arg\max_{c=1,..C} \{f_T^c(v_i)\}.$$

The number of steps T is set to be large enough such that f_T^c is close to its limit $f_\infty^c := \lim_{T\to\infty} f_T^c$. If G is connected, it can be shown that f_∞^c is independent of the choice of f_0^c. Even if G is disconnected, the algorithm can be performed on each connected component separately and again the limit f_∞^c for each component will be independent of the choice of f_0^c.

We will now adapt the label propagation algorithm to higher dimensional walks, namely, walks on oriented edges. Given any random walk on the set of oriented edges (and an absorbing death state Θ), its probability transition matrix P could be used to propagate labels in the same manner as the above algorithm. However, this will treat and label the two orientations of a single edge separately as though they are unrelated. As found in this chapter and in Parzanchevski and Rosenthal (2012), geometric meaning and interesting long-term behavior is obtained by transforming and normalizing P into a normalized propagation matrix, and applying it not to functions on the state space but to 1-cochains. In this way we will infer only one label per edge. One major change, however, is that labels will become oriented themselves. That is, given an oriented edge e_+ and a class c, the propagation algorithm may assign a positive confidence that e_+ belongs to class c or a negative confidence that e_+ belongs to class c, which we view as a positive confidence that e_+ belongs to class $-c$ or, equivalently, that e_- belongs to class c. This construction applies to systems in which every class has two built-in orientations or signs, or the class information has a directed sense of "flow".

For example, imagine water flowing along a triangle complex in two dimensions. Given an oriented edge, the water may flow in the positive or negative direction along the edge. A "negative" flow of water in the direction of e_+ can be interpreted as a positive flow in the direction of e_-. Perhaps the flow along a few edges is observed and one wishes to infer the direction of the flow along all the other edges. Unlike in the graph case, a single class of flow already presents a classification challenge. Or consider multiple streams of water colored according to the C classes, we may want to know which stream dominates the flow along each edge and in which direction. In order to make these inferences, it is necessary to make some assumption about how labels should propagate from one edge to the next. When considering water flow, it is intuitive to make the following two assumptions.

1. **Local Consistency of Motion.** If water is flowing along an oriented edge

$[v_i, v_j]$ in the positive direction, then for every triangle $[v_i, v_j, v_k]$ the water should also tend to flow along $[v_i, v_k]$ and $[v_k, v_j]$ in the positive directions.

2. **Preservation of Mass.** The total amount of flow into and out of each vertex (along edges connected to the vertex) should be the same.

In fact, either one of these assumptions is sufficient to infer oriented class labels given the observed flow on a few edges. Depending on which assumptions one chooses, different normalized propagation matrices $\widetilde{A}_{L,p}$ (see section 4.5) may be applied. For example, $L = L_1^{\text{up}}$ will enforce local consistency of motion without regard to preservation of mass, while $L = L_1^{\text{down}}$ will do the opposite. A reasonable way of preserving both assumptions is by using $L = L_1$ as shown in Example 42.

We now state a simple algorithm, analogous to the one for graphs, that propagates labels on edges to infer a partially-observed flow. Let X be a simplicial complex of dimension $d \geq 1$ and let $X_+^1 = \{e_1, \ldots, e_n\}$ be a choice of orientation for the set of edges. Without loss of generality, assume that oriented edges $e_u + 1, \ldots, e_{n=u+\ell}$ have been classified with class c (not $-c$). Similar to the graph case, we apply a recursive label propagation process to an initial distribution vector $f_0^c : X_+^1 \to \mathbb{R}$ measuring the *a priori* confidence that each oriented edge is in class c. See Algorithm 1 for the procedure. The result of the algorithm is a set of estimates of the relative confidence that each edge is in class c with some orientation.

After running the algorithm, an unlabelled edge e_i is assigned the oriented class $\operatorname{sgn}(f_T^c(e_i))c$ where $c = \arg\max_{c=1,..C}\{|f_T^c(e_i)|\}$. We now prove that given enough iterations T the algorithm converges and the resulting assigned labels are meaningful. The proof uses the same methods as the one found in Zhu et al. (2005) for the graph case.

Proposition 40. *Using the notation of section 4.5, assume that L is a symmetric X_+^k-matrix with Spec$(LD_L^{-1}) \subset [0, \Lambda]$. Let $\widetilde{A}_{L,p}$ be the normalized p-lazy propagation*

Algorithm 1: Edge propagtion algorithm.

Data: Simplicial complex X, set of oriented edges

$$X_+^1 = \{e_1, \dots, e_u, e_{u+1}, ..., e_{u+\ell}\}$$

with $e_{u+1}, \dots, e_{u+\ell}$ labelled with oriented classes $\pm 1, .., \pm C$, initial distribution vector $f_0^c : X_+^1 \to \mathbb{R}$, number of iterations T

Result: Confidence of class membership and direction for unlabelled edges $\{f_*^c(e_1), ..., f_*^c(e_u)\}_{c=1}^C$

for $c = 1$ *to* C **do**
  for $t = 1$ *to* T **do**
    $f_t^c \leftarrow \widetilde{A}_{L,p} f_{t-1}^c$;
    $f_t^c(e_i) \leftarrow 1$ for e_i labelled with class c;
    $f_t^c(e_i) \leftarrow -1$ for e_i labelled with class $-c$
  end
end
$\{f_*^c(e_1), ..., f_*^c(e_u)\}_{c=1}^C \leftarrow \{f_T^c(e_1), ..., f_T^c(e_u)\}_{c=1}^C$;

matrix as defined in Definition 35. If $\frac{\Lambda-2}{2K+\Lambda-2} < p < 1$ *and if no vector in* $\ker L$ *is supported on the set of unclassified edges, then Algorithm 1 converges. That is,*

$$\lim_{T\to\infty} f_T^c =: f_\infty^c = \begin{pmatrix} \psi^c \\ (I - A_4)^{-1} A_3 \psi^c \end{pmatrix},$$

where A_4 *and* A_3 *are submatrices of* $\widetilde{A}_{L,p}$ *and* ψ^c *is the class function on edges labelled with* $\pm c$ *(for which* $\psi^c(e_i) = \pm 1$*). In addition,* f_∞^c *depends neither on the initial distribution* f_0^c *nor on the lazy probability* p.

Proof. First, note that we are only interested in the convergence of $f_T^c(e_i)$ for e_i not labelled $\pm c$. Partition f_T^c and $\widetilde{A}_{L,p}$ according to whether e_i is labelled $\pm c$ or not as

$$f_T^c = \begin{pmatrix} \psi^c \\ \hat{f}_T^c \end{pmatrix} \qquad \text{and} \qquad \widetilde{A}_{L,p} = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}.$$

The recursive definition of f_T^c in Algorithm 1 can now be rewritten as $\hat{f}_T^c = A_4 \hat{f}_{T-1}^c + A_3 \psi^c$. Solving for $\hat{f}_T^c$ in terms of $\hat{f}_0^c$ yields

$$\hat{f}_T^c = (A_4)^k \hat{f}_0^c + \sum_{i=0}^{T-1} (A_4)^i A_3 \psi^c.$$

In order to prove convergence of $\hat{f}_T^c$, it suffices to prove that A_4 has only eigenvalues strictly less than 1 in absolute value. This ensures that $(A_4)^k \hat{f}_0^c$ converges to zero (eliminating dependence on the initial distribution) and that $\sum_{i=0}^{k-1} (A_4)^i A_3 \psi^c$ converges to $(I - A_4)^{-1} A_3 \psi^c$ as $k \to \infty$. We will prove that $\mathrm{Spec}(A_4) \subset (-1, 1)$ by relating $\mathrm{Spec}(A_4)$ to $\mathrm{Spec}(LD_L^{-1}) \subset [0, \Lambda]$ as follows.

First, partition L and D_L similar to $\widetilde{A}_{L,p}$ as

$$L = \begin{pmatrix} L_1 & L_2 \\ L_3 & L_4 \end{pmatrix} \qquad \text{and} \qquad D_L = \begin{pmatrix} D_1 & 0 \\ 0 & D_4 \end{pmatrix}.$$

so that

$$A_4 = I - \frac{1-p}{p(K-1)+1} L_4 D_4^{-1}.$$

Hence $\mathrm{Spec}(A_4)$ is determined by $\mathrm{Spec}(L_4 D_4^{-1})$, or to be more specific, $\lambda \in \mathrm{Spec}(L_4 D_4^{-1}) \Leftrightarrow 1 - \frac{1-p}{p(K-1)+1} \lambda \in \mathrm{Spec}(A_4)$. Furthermore, note that $L_4 D_4^{-1}$ and $D_4^{-1/2} L_4 D_4^{-1/2}$ are similar matrices and share the same spectrum. It turns out that the spectrum of $D_4^{-1/2} L_4 D_4^{-1/2}$ is bounded within the spectrum of $D_L^{-1/2} L D_L^{-1/2}$, which in turn is equal to $\mathrm{Spec}(LD_L^{-1}) \subset [0, \Lambda]$ by similarity. Let g be an eigenvector of $D_4^{-1/2} L_4 D_4^{-1/2}$ with eigenvalue λ and let $g_1, \ldots, g_j$ be an orthonormal basis of eigenvectors of $D_L^{-1/2} L D_L^{-1/2}$ (such a basis exists since it is a symmetric matrix) with eigenvalues $\mu_1, \ldots, \mu_j$. We can write

$$\begin{pmatrix} \mathbf{0}_c \\ g \end{pmatrix} = \alpha_1 g_1 + \ldots + \alpha_j g_j$$

for some $\alpha_1, \ldots, \alpha_j$, where $\mathbf{0}_c$ is the vector of zeros with length equal to the number of edges classified as $\pm c$. Then

$$\begin{aligned}
\alpha_1\mu_1 g_1 + \ldots + \alpha_j\mu_j g_j &= D_L^{-1/2} L D_L^{-1/2} \begin{pmatrix} \mathbf{0}_c \\ g \end{pmatrix} \\
&= \begin{pmatrix} D_1^{-1/2} L_1 D_1^{-1/2} & D_1^{-1/2} L_2 D_4^{-1/2} \\ D_4^{-1/2} L_3 D_1^{-1/2} & D_4^{-1/2} L_4 D_4^{-1/2} \end{pmatrix} \begin{pmatrix} \mathbf{0}_c \\ g \end{pmatrix} \\
&= \begin{pmatrix} D_1^{-1/2} L_2 D_4^{-1/2} g \\ D_4^{-1/2} L_4 D_4^{-1/2} g \end{pmatrix} \\
&= \begin{pmatrix} D_1^{-1/2} L_2 D_4^{-1/2} g \\ \lambda g \end{pmatrix}.
\end{aligned}$$

Taking the Euclidean norm of the beginning and ending expressions, we see that

$$\begin{aligned}
|\alpha_1\mu_1| + \ldots + |\alpha_j\mu_j| &= \left\| \begin{pmatrix} D_1^{-1/2} L_2 D_4^{-1/2} g \\ \lambda g \end{pmatrix} \right\|_2 \\
&\geq \|\lambda g\|_2 \\
&= \lambda(|\alpha_1| + \ldots + |\alpha_j|).
\end{aligned}$$

Because we assumed that $\mu_i \in [0, \Lambda]$ for all i, it would be a contradiction if $\lambda < 0$ or $\lambda > \Lambda$. The case $\lambda = 0$ is possible if and only if there is a vector in $\ker L$ that is supported on the unlabelled edges. To see this, note that if $\lambda = 0$ then

$$\begin{aligned}
\alpha_1^2\mu_1 + \ldots + \alpha_j^2\mu_j &= \begin{pmatrix} \mathbf{0}_c \\ g \end{pmatrix}^T D_1^{-1/2} L_2 D_4^{-1/2} \begin{pmatrix} \mathbf{0}_c \\ g \end{pmatrix} \\
&= \begin{pmatrix} \mathbf{0}_c \\ g \end{pmatrix}^T \begin{pmatrix} D_1^{-1/2} L_2 D_4^{-1/2} g \\ \lambda g \end{pmatrix} \\
&= 0
\end{aligned}$$

which implies $\alpha_i\mu_i = 0$ for all i and therefore $\left(\begin{smallmatrix} \mathbf{0}_c \\ g \end{smallmatrix}\right) \in \ker L$. Finally, since we assumed that no vector in $\ker L$ is supported on the unlabelled edges and that $\frac{\Lambda-2}{2K+\Lambda-2} < p < 1$, we conclude that $\mathrm{Spec}(L_4 D_4^{-1}) \subset (0, \Lambda]$ and therefore $\mathrm{Spec}(A_4) \subset \left[1 - \frac{1-p}{p(K-1)+1}\Lambda, 1\right) \subset (-1, 1)$.

To see that the solution $\hat{f}^c_\infty = (I - A_4)^{-1} A_3 \psi^c$ does not depend on p, note that $I - A_4$ is a submatrix of $\frac{1-p}{p(K-1)+1} L D_L^{-1}$ so that $\frac{p(K-1)+1}{1-p}(I - A_4)$ does not depend on p. Then write $\hat{f}^c_\infty$ as

$$\hat{f}^c_\infty = \left[\frac{p(K-1)+1}{1-p}(I - A_4)\right]^{-1} \times \frac{1}{1-p} A_3 \psi^c$$

and note that $\frac{p(K-1)+1}{1-p} A_3$ is an *off-diagonal* submatrix of $\frac{p(K-1)+1}{1-p} I - L D_L^{-1}$ and therefore does not depend on p either. □

Note that while the limit f^c_∞ exists, the matrix $I - A_4$ could be ill-conditioned. In practice, it may be better to approximate f^c_∞ with f^c_t for large enough t. Also, the algorithm will converge faster for smaller values of p and if $\hat{f}^c_0 = \mathbf{0}$.

4.6.3 Experiments

We use some simulations to illustrate how Algorithm 1 works.

Example 41. *Figure 4.2 shows a simplicial complex in which a single oriented edge e_1 has been labelled with class c (indicated by the red color) and all other edges are unlabelled. Figure 4.3 shows what happens when this single label is propagated $T = 100$ steps using Algorithm 1 with $L = L_1^{up}$, $p = 0.9$, and with f^c_0 equal to the indicator function on e_1. After the T steps have been performed the edges are oriented and labelled according to the sign of f^c_k (if $f^c_k(e_i) = 0$ for an oriented edge e_i, then that edge is left unoriented and unlabelled in the figure). Figures 4.4 and 4.5 show the same thing with $L = L_1^{down}$ and $L = L_1$, respectively. The results using L_1^{up} and L_1^{down} have a clear resemblance to magnetic fields. When $L = L_1^{down}$, "mass" is preserved which creates multiple vortices where the flow spins around a triangle. The walk using L_1^{up} tries to maintain local consistency of motion, creating sources and sinks in the process. The full L_1 walk strikes somewhat of a balance between the two, resulting in a more circular flow with a single vortex in the lower left.*

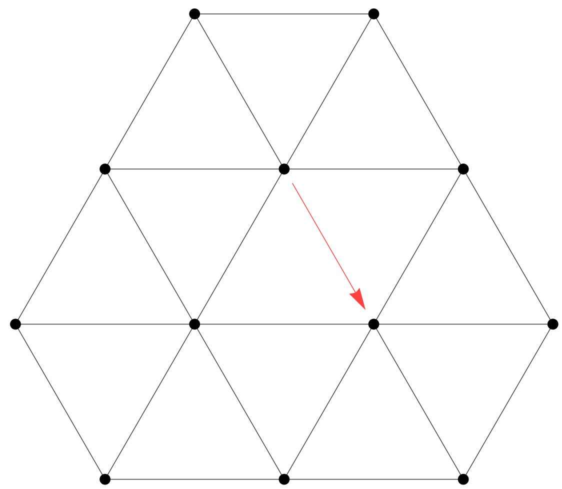

FIGURE 4.2: A 2-complex with a labelled edge.

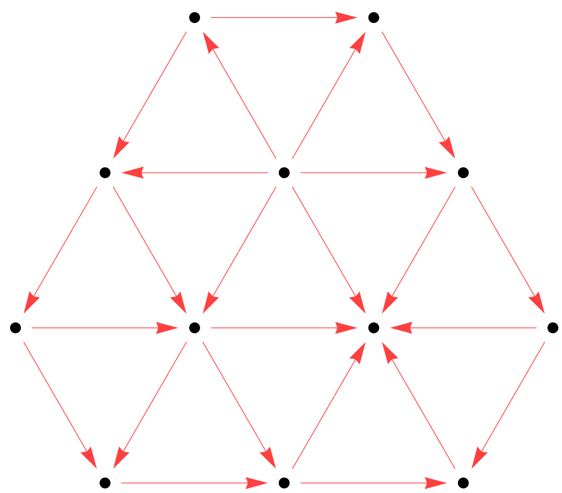

FIGURE 4.3: Label propagation with $L = L_1^{\mathrm{up}}$.

Example 42. *Figure 4.6 shows a simplicial complex in which two edges have been labelled with class* $c = 1$ *(indicated by the red color) and two more edges have been labelled with class* $c = 2$ *(indicated by the blue color). Figure 4.7 shows what happens when the labels are propagated* $T = 1000$ *steps using Algorithm 1 with* $L = L_1$*,* $p = 0.9$*, and* f_0^c *equal to the indicator function on the oriented edges labelled with classes* $c = 1, 2$*. Every edge is then oriented and labelled according to the sign of* $f_T^{c=1}$*, if* $|f_T^{c=1}| > |f_T^{c=2}|$*, or* $f_T^{c=2}$*, if* $|f_T^{c=1}| < |f_T^{c=2}|$*. Notice that only a small number of labels are needed to induce large-scale circular motion. Near the middle, a few*

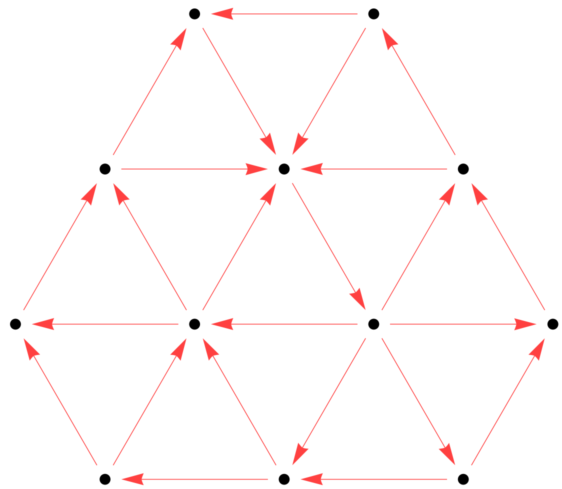

FIGURE 4.4: Label propagation with $L = L_1^{\mathrm{down}}$.

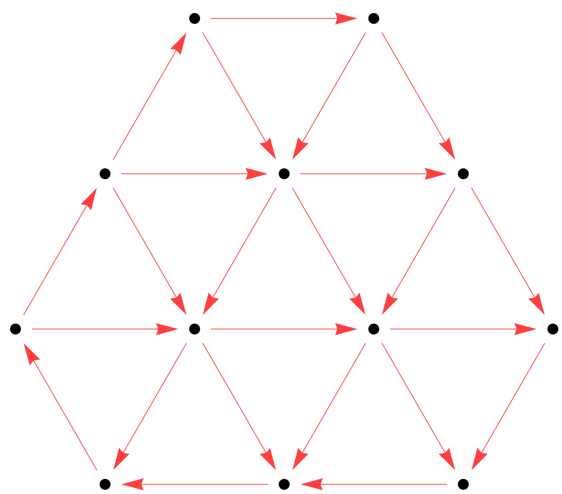

FIGURE 4.5: Label propagation with $L = L_1$.

blue labels mix in with the red due to the asymmetry of the initial labels.

4.7 Discussion

In this chapter, we introduced a random walk with absorbing states on simplicial complexes. Given a simplicial complex of dimension d, the relation between the random walk and the spectrum of the k-dimensional Laplacian for $1 \leq k \leq d$ was examined. We compared the Dirichlet random walk we introduced to the Neumann random

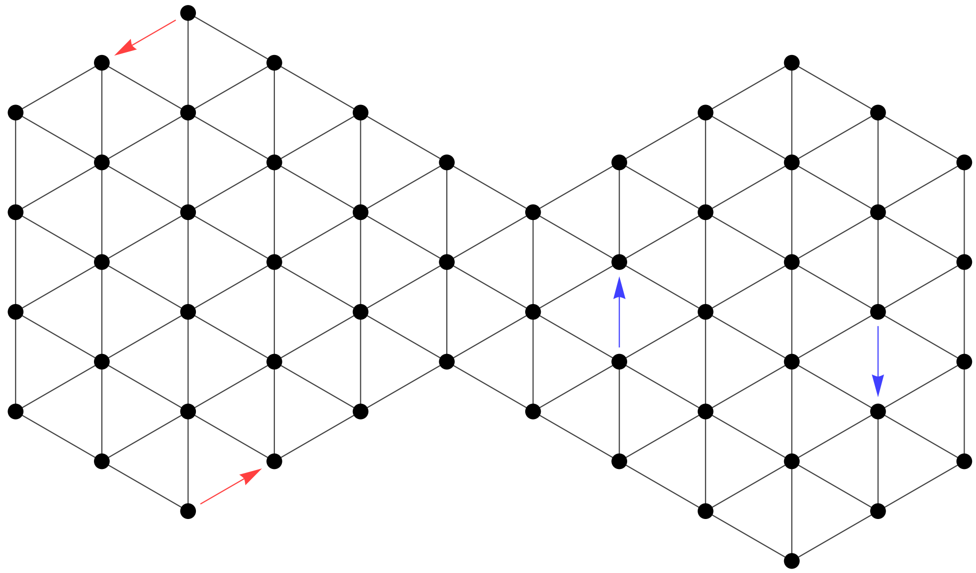

FIGURE 4.6: A 2-complex with two different labels on four edges.

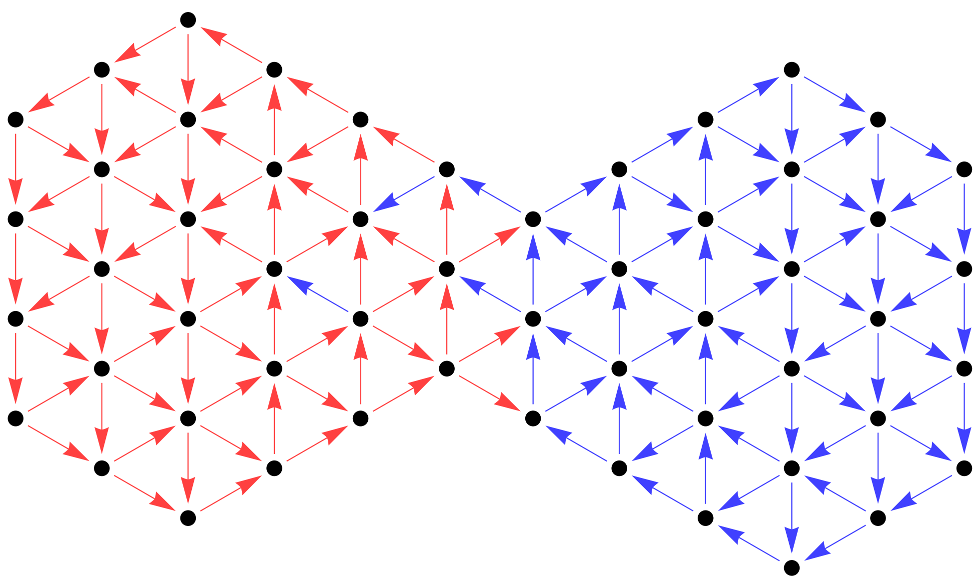

FIGURE 4.7: Label propagation with $L = L_1$.

walk introduced by Rosenthal and Parzanchevski in Parzanchevski and Rosenthal (2012).

There remain many open questions about random walks on simplicial complexes and the spectral theory of higher order Laplacians. Possible future directions of research include:

(1) Is there a Brownian process on a manifold that corresponds to the continuum limit of these new random walks?

(2) Is it possible to use conditioning techniques from stochastic processes such as Doob's h-transform to analyze these walks?

(3) What applications do these walks have to problems in machine learning and statistics?

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Biography

The author, John Joseph Steenbergen, was born on July 29, 1985 in Indianapolis, IN. John received a B.S. in Math (Honors) and a B.S. in Statistics from Purdue University in May 2008, a M.S. in Mathematics from Duke University in December 2010, and a PhD in Mathematics from Duke University in December 2013. John held a Duke Endowment fellowship to support his graduate studies at Duke University from the fall of 2008 up until the fall of 2012. He expects to be working as a Research Assistant Professor at the Mathematics, Statistics, and Computer Science department of the University of Illinois at Chicago in the spring semester of 2014.